TIGHT PERFORMANCE BOUNDS FOR DISTRIBUTED DETECTION

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ABSTRACT

Evaluating the performance measures of distributed detection in sensor networks is important for design procedures aiming at optimal configurations. Direct computation of the global error probabilities is a difficult problem and feasible only in some special cases. In this paper, we present an approach that provides closed-form upper bounds on the detection error probabilities in the parallel fusion network which are both computationally simple and numerically tight. The bounds are derived by combining a probability inequality formulated by Hoeffding with a multiplicative form factor which is due to Talagrand. We demonstrate that the bounds apply to sensor networks of varying size, an arbitrary number of local quantization levels, and non-identical sensors.

Index Terms— Distributed detection, parallel fusion network, quantization, error analysis, performance bounds

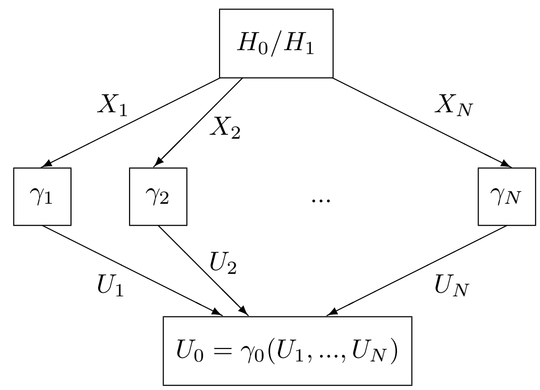

Fig. 1. Parallel fusion network.

1. INTRODUCTION

System-wide optimization of sensor networks for detection applications requires efficient methods for evaluating the performance measures of interest, e.g., the probability of false alarm, the probability of miss, or the average probability of error. Direct computation of these probabilities is feasible only for networks with a small number of sensors and few quantization levels. Existing approaches to performance evaluation which circumvent direct computation include determining the asymptotic error exponents of the distributed detection system as the number of sensor nodes tends to infinity [1, 2]. In [3], Aldosari and Moura have presented an application of the saddlepoint method to provide computationally simple and accurate approximations for the various error probabilities. However, the obtained expressions require the numerical solution of a saddlepoint equation. In this paper, we present closed-form upper bounds on the detection error probabilities for sensor networks of arbitrary size which are both computationally simple and numerically tight.

We consider the parallel fusion network in which all sensors process their observations independently and transmit local decisions to a fusion center for decision combining. Optimal fusion of local decisions in a Bayesian framework yields expressions for the global detection error probabilities, i.e., the probability of false alarm and the probability of miss, taking the form of tail probabilities. Our aim is to bound these probabilities by analytical expressions which are both computationally inexpensive and tight. For this purpose, we adopt a probability inequality introduced by Hoeffding which provides exponential bounds on the tail probabilities for sums of bounded random variables [4]. In order to improve the sharpness of the obtained bounds, we employ a multiplicative form factor following a technique developed by Talagrand [5]. Finally, it turns out that in sensor network scenarios the form factor can be simplified considerably leading to computationally inexpensive expressions.

The remainder of the paper is organized as follows. In section 2, the problem of distributed detection in the parallel fusion network is stated. In section 3, we formulate the Bayes-optimal fusion rule and give the distribution of the involved random variables. Expressions for the global detection error probabilities are discussed in section 4. The main results are presented in section 5, where we derive computationally simple and tight analytical bounds on the global detection error probabilities. We present numerical results in section 6 and conclude in section 7.

2. DISTRIBUTED DETECTION

The problem of distributed detection in the parallel fusion network is as follows. We consider a binary hypothesis testing problem with hypotheses H_0 and H_1 describing the state of the observed environment. The associated prior probabilities are $\pi_0 = P(H_0)$ and $\pi_1 = P(H_1)$. A set of N sensors receive random observations $X_i \in \mathcal{X}_i$, $i = 1, ..., N$, which are assumed to be conditionally independent across sensors given the underlying hypothesis, i.e., the joint conditional probability density function (pdf) of the observations factorizes to

$$f(x_1, ..., x_N|H_k) = \prod_{i=1}^{N} f_i(x_i|H_k), \quad k = 0, 1.$$

The sensors process their observations independently by forming local decisions $U_i = \gamma_i(X_i)$, $i = 1, ..., N$. In the case of M-ary quantization, the sensor decision rules γ_i are mappings

$$\gamma_i : \mathcal{X}_i \rightarrow \{1, ..., M\}, \quad i = 1, ..., N.$$

Warren and Willet have shown that the sensor decision rules leading to globally optimal configurations under the Bayes criterion are monotone likelihood ratio partitions of the sensor observation spaces $\mathcal{X}_1, ..., \mathcal{X}_N$, provided that the observations are conditionally independent across sensors [6]. Hence, in the optimal design of distributed detection systems under the assumption of conditional independence, it is necessary only to consider sensor decision rules γ_i which can be parameterized by a set of real quantization thresholds $\{\lambda_{i,0}, ..., \lambda_{i,M}\}$, where $\lambda_{i,0} = 0$, $\lambda_{i,M} = \infty$, and $\lambda_{i,j} \leq \lambda_{i,j+1}$. In this way, each local decision U_i is characterized by two sets of conditional probabilities $\{\alpha_{i1}, ..., \alpha_{iM}\}$ and $\{\beta_{i1}, ..., \beta_{iM}\}$ with

$$\alpha_{ij} \triangleq P(U_i = j|H_0) = P(\lambda_{i,j-1} \leq L_i < \lambda_{i,j}|H_0),$$
$$\beta_{ij} \triangleq P(U_i = j|H_1) = P(\lambda_{i,j-1} \leq L_i < \lambda_{i,j}|H_1),$$

where $L_i = f_i(X_i|H_1)/f_i(X_i|H_0)$ is the local likelihood ratio of observation X_i. In the sequel, we assume that the conditional probabilities $\{\alpha_{i1}, ..., \alpha_{iM}\}$ and $\{\beta_{i1}, ..., \beta_{iM}\}$ are computable given the local observation statistics $f_i(\cdot|H_k)$, $k = 0, 1$, and the quantization thresholds $\{\lambda_{i,0}, ..., \lambda_{i,M}\}$ for all $i = 1, ..., N$.

In this paper, we assume error-free communication links between the sensors and the fusion center. The sensors transmit local decisions $U_1, ..., U_N$ to the fusion center which combines them to yield the global decision $U_0 = \gamma_0(U_1, ..., U_N)$. The fusion rule γ_0 is a binary-valued mapping

$$\gamma_0 : \{1, ..., M\}^N \rightarrow \{0, 1\}.$$

The sensor network detection performance is characterized by the global probability of false alarm $P_f = P(U_0 = 1|H_0)$ and the global probability of miss $P_m = P(U_0 = 0|H_1)$. The average probability of error $P_e = \pi_0 P_f + \pi_1 P_m$ is a weighted sum of the false alarm and miss rate.

3. OPTIMAL FUSION

We consider optimal fusion of the local decisions $U_1, ..., U_N$ in a Bayesian framework with a zero-one loss function. Thus, the objective is to determine the fusion rule that minimizes the average probability of error. According to Varshney [7], the optimal fusion rule under the Bayes risk criterion in the case of conditionally independent decisions can be performed by evaluating a log-likelihood ratio test of the form

$$\sum_{i=1}^{N} \mathcal{L}_i \underset{0}{\overset{1}{\gtrless}} \log\left(\frac{\pi_0}{\pi_1}\right) \triangleq \vartheta, \tag{1}$$

where $\mathcal{L}_i = \log(P(U_i|H_1)/P(U_i|H_0))$ is the log-likelihood ratio of making a decision U_i, and ϑ is the fusion threshold. The log-likelihood ratio $\mathcal{L}_i$ is a discrete random variable that takes one out of M possible values

$$l_{ij} \triangleq \log\left(\frac{\beta_{ij}}{\alpha_{ij}}\right), \quad j = 1, ..., M, \tag{2}$$

and has conditional probability mass functions given by

$$P(\mathcal{L}_i = l_{ij}|H_0) = \alpha_{ij}, \tag{3}$$
$$P(\mathcal{L}_i = l_{ij}|H_1) = \beta_{ij}. \tag{4}$$

For equations (3) and (4) to hold, we assume that all possible realizations of the log-likelihood ratio $\mathcal{L}_i$ in (2) are distinct.

4. DETECTION ERROR PROBABILITIES

When using the Bayes-optimal fusion rule according to (1), the global probability of false alarm P_f and the global probability of miss P_m are determined by the conditional tail probabilities

$$P_f = P(\sum_{i=1}^{N} \mathcal{L}_i \geq \vartheta|H_0) \tag{5}$$

and

$$P_m = P(\sum_{i=1}^{N} \mathcal{L}_i < \vartheta|H_1). \tag{6}$$

Exact computation of these tail probabilities can be performed by evaluating the test (1) for all M^N possible realizations of the local decisions $U_1, ..., U_N$ under each hypothesis and summing up the corresponding probabilities. This approach is not appropriate for large-scale sensor networks, especially when multiple evaluations of the detection performance measures are necessary for sensor network optimization.

5. PERFORMANCE BOUNDS

In this section, we apply a combination of a probability inequality formulated by Hoeffding and a method developed by Talagrand which delivers a multiplicative form factor in order to obtain sharp bounds on the tail probabilities (5) and (6).

5.1. Probability of false alarm

In this subsection, we derive an upper bound on the global probability of false alarm $P_f = P(U_0 = 1|H_0)$. Introducing the conditional zero-mean random variables

$$\widehat{\mathcal{L}}_i \triangleq \mathcal{L}_i - \mathbb{E}[\mathcal{L}_i|H_0] = \mathcal{L}_i - \sum_{j=1}^{M} \alpha_{ij} l_{ij}, \quad i = 1, ..., N,$$

where $\mathbb{E}[\cdot|H_k]$ denotes conditional expectation given hypothesis H_k, and the new threshold

$$\vartheta_0 \triangleq \vartheta - \sum_{i=1}^{N}\sum_{j=1}^{M} \alpha_{ij} l_{ij},$$

we obtain

$$P_f = P(\sum_{i=1}^{N} \widehat{\mathcal{L}}_i \geq \vartheta_0 | H_0). \tag{7}$$

Considering equation (7), we first apply a probability inequality for the sum of zero-mean bounded random variables formulated by Hoeffding [4]. In order to improve the sharpness of the obtained exponential bound, we use a method introduced by Talagrand which delivers a multiplicative form factor [5]. Eventually, we obtain the bound

$$P_f \leq \left(\varphi\left(\frac{\vartheta_0}{\sigma_0\sqrt{N}}\right) + \frac{K_0 B_0}{\sigma_0\sqrt{N}} \right) e^{-NH(\sigma_0^2, b_0, \frac{\vartheta_0}{N})}, \tag{8}$$

where $\varphi(x) = e^{\frac{x^2}{2}}(1 - \Phi(x))$, and Φ is the cumulative distribution function (cdf) of the standard normal distribution. The quantities involved are given by

$$\sigma_0^2 \triangleq \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{M}(l_{ij} - \sum_{k=1}^{M}\alpha_{ik} l_{ik})^2 \cdot \alpha_{ij},$$

$$b_0 \triangleq \max_{i,j}\ l_{ij} - \sum_{k=1}^{M} \alpha_{ik} l_{ik},$$

$$B_0 \triangleq \max_{i,j}\ |l_{ij} - \sum_{k=1}^{M} \alpha_{ik} l_{ik}|,$$

and K_0 is a small constant value which has to be obtained numerically according to [5]. The function H is defined as

$$H(\sigma^2, b, t) \triangleq \left(1 + \frac{bt}{\sigma^2}\right)\frac{\sigma^2}{b^2 + \sigma^2}\log\left(1 + \frac{bt}{\sigma^2}\right) + \left(1 - \frac{t}{b}\right)\frac{b^2}{b^2+\sigma^2}\log\left(1 - \frac{t}{b}\right).$$

5.2. Probability of miss

The construction of an upper bound on the global probability of miss $P_m = P(U_0 = 0|H_1)$ follows the same lines as for the probability of false alarm. We obtain

$$P_m \leq \left(\varphi\left(\frac{\vartheta_1}{\sigma_1\sqrt{N}}\right) + \frac{K_1 B_1}{\sigma_1\sqrt{N}} \right) e^{-NH(\sigma_1^2, b_1, \frac{\vartheta_1}{N})}, \tag{9}$$

with the functions φ and H as defined above and the corresponding quantities

$$\vartheta_1 \triangleq \sum_{i=1}^{N}\sum_{j=1}^{M} \beta_{ij} l_{ij} - \vartheta,$$

$$\sigma_1^2 \triangleq \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{M}(l_{ij} - \sum_{k=1}^{M}\beta_{ik} l_{ik})^2 \cdot \beta_{ij},$$

$$b_1 \triangleq \max_{i,j}\ \sum_{k=1}^{M} \beta_{ik} l_{ik} - l_{ij},$$

$$B_1 \triangleq \max_{i,j}\ |l_{ij} - \sum_{k=1}^{M} \beta_{ik} l_{ik}|.$$

Again, K_1 has to be obtained numerically.

5.3. Approximate performance bounds

Due to the fact that $K_0, K_1 \ll \sqrt{N}$ for typical sensor network scenarios, we will drop the second term of the multiplicative form factor in (8) and (9) and consider the approximate performance bounds

$$P_f \lesssim \varphi\left(\frac{\vartheta_0}{\sigma_0\sqrt{N}}\right) e^{-NH(\sigma_0^2, b_0, \frac{\vartheta_0}{N})}, \tag{10}$$

$$P_m \lesssim \varphi\left(\frac{\vartheta_1}{\sigma_1\sqrt{N}}\right) e^{-NH(\sigma_1^2, b_1, \frac{\vartheta_1}{N})}. \tag{11}$$

The evaluation of these expressions is straightforward and yields tight bounds on the corresponding error probabilities, which is illustrated by numerical examples in the next section.

6. NUMERICAL RESULTS

We indicate the sharpness of the expressions (10) and (11) by comparing the bound on the average probability or error

$$P_e \lesssim \pi_0 \varphi\left(\frac{\vartheta_0}{\sigma_0\sqrt{N}}\right) e^{-NH(\sigma_0^2, b_0, \frac{\vartheta_0}{N})} + \pi_1 \varphi\left(\frac{\vartheta_1}{\sigma_1\sqrt{N}}\right) e^{-NH(\sigma_1^2, b_1, \frac{\vartheta_1}{N})} \tag{12}$$

with the sensor network detection performance obtained by extensive Monte Carlo simulations.

Example 1. In the first example, we assume binary quantization of local sensor observations, i.e., $M = 2$. We assume that the hypotheses H_0 and H_1 have prior probabilities $\pi_0 = 0.8$ and $\pi_1 = 0.2$. The local detectors transmit realizations of $0/1$-valued random variables $U_1, ..., U_N$ which shall be characterized by the following local false alarm and miss probabilities

$$\alpha_{i1} = P(U_i = 1|H_0) = 0.2 + (i-1)\cdot 0.002,$$
$$\beta_{i0} = P(U_i = 0|H_1) = 0.5 - i \cdot 0.002, \quad i = 1, ..., N.$$

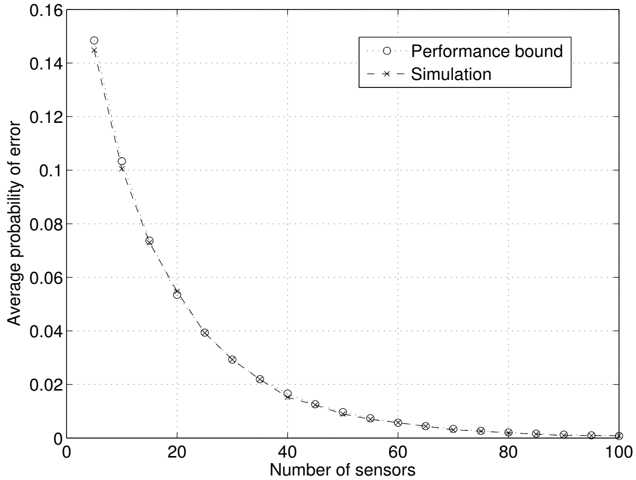

Fig. 2. Performance bound vs. simulation ($M = 2$).

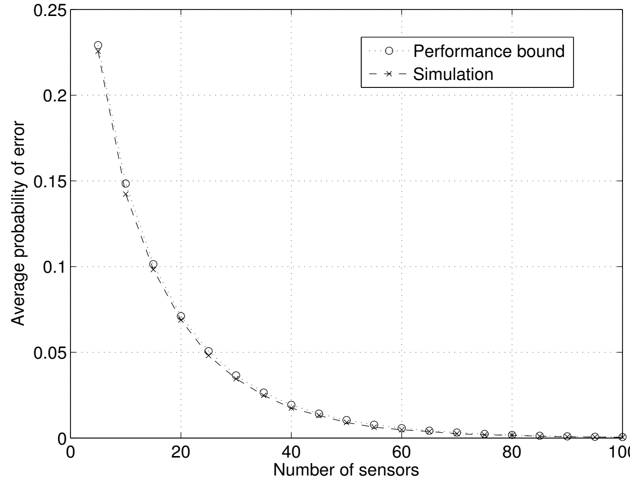

Fig. 3. Performance bound vs. simulation ($M = 4$).

Reasonably, the higher the false alarm rate of a sensor the lower shall be its miss rate. The number of sensors N varies between 5 and 100. Fig. 2 shows the evaluation of the performance bound (12) on the average probability of error in comparison with numerical results obtained by extensive Monte Carlo simulations. Besides its high accuracy, the performance bound appears to be valid for both small and large sensor network scenarios.

Example 2. Here, we assume a network of N quaternary local detectors, i.e., $M = 4$. Now we assume that the hypotheses H_0 and H_1 are equally likely to occur, i.e., $\pi_0 = \pi_1 = 1/2$. We specify the conditional probabilities of the local decisions U_i under hypotheses H_0 and H_1 in the following way

$$\alpha_{i1} = P(U_i = 1|H_0) = 0.3, \quad \alpha_{i2} = P(U_i = 2|H_0) = 0.3,$$
$$\alpha_{i3} = P(U_i = 3|H_0) = 0.2, \quad \alpha_{i4} = P(U_i = 4|H_0) = 0.2,$$
$$\beta_{i1} = P(U_i = 1|H_1) = 0.1, \quad \beta_{i2} = P(U_i = 2|H_1) = 0.2,$$
$$\beta_{i3} = P(U_i = 3|H_1) = 0.3, \quad \beta_{i4} = P(U_i = 4|H_1) = 0.4,$$

for all $i = 1, ..., N$, i.e., in this example we assume identical local quantizers. Again, we consider sensor networks consisting of 5 to 100 sensors. Fig. 3 shows numerical results provided by the performance bound (12) and extensive simulations. As in the previous example, the comparison reveals the high accuracy and validity of the constructed performance bound.

7. CONCLUSIONS

We presented an approach to construct tight bounds on the detection error probabilities in parallel fusion sensor networks. The results in this paper provide computationally simple expressions for performance bounds that apply to sensor networks of varying size with an arbitrary number of quantization levels and non-identical sensors. The bounds are supposed to be especially useful in sensor network optimization, where multiple evaluations of the performance measures of interest are necessary. Furthermore, the obtained analytical expressions pave the way for optimization by setting the derivatives of the performance bounds with respect to selected system parameters equal to zero.

8. REFERENCES

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