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Eigenvalue-Based Optimum-Power Allocation for Gaussian Vector Channels

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***Abstract*—In this correspondence, we determine the optimal power allocation to antennas in a Gaussian vector channel subject to ℓ_p-norm constrained eigenvalues. Optimal solutions are characterized by using directional derivatives of the mutual information. As the central result, the optimal power assignment is obtained as the level crossing points of a set of simple monotone functions. The well-known water-filling principle for sum power constraints is retrieved as the limiting case $p = 1$. A nested Newton type algorithm is given for finding numerical solutions.**

***Index Terms*—Capacity, concavity, directional derivatives, generalized water-filling ℓ_p-norms, max and sum power constraints, multiple-input–multiple-output (MIMO).**

I. Introduction and System Model

Power allocation is an important challenge when transmitting over multiple antennas. The general model we adopt for this purpose is a linear vector channel with Gaussian noise and arbitrary input distribution; in standard notation

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}. \tag{1}$$

The complex $r \times t$ matrix $\boldsymbol{H}$ describes the linear transformation the signal undergoes during transmission. The random noise vector $\boldsymbol{n} \in \mathbb{C}^r$ is circularly symmetric complex Gaussian distributed (see [1]) with expectation $\mathbf{0}$ and covariance matrix $\mathbb{E}(\boldsymbol{n}\boldsymbol{n}^*) = \boldsymbol{I}_r$, denoted by $\boldsymbol{n} \sim \mathrm{SCN}(\mathbf{0}, \boldsymbol{I}_r)$. The complex zero mean input vector $\boldsymbol{x}$ is subject to power constraints described by

$$\mathbb{E}(\boldsymbol{x}\boldsymbol{x}^*) = \boldsymbol{Q} \in \mathcal{Q} \tag{2}$$

for some set of nonnegative definite matrices $\mathcal{Q}$. Finally, $\boldsymbol{y}$ denotes the received vector.

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The general model applies to many different communication systems, including a point to point multiple-input–multiple-output (MIMO) transmission system with r receive antennas and t transmit antennas (see [1]–[3]). Transmit beamforming, broadcast and multiple access channels, cellular code-division multiple-access (CDMA) radio, and digital wireline systems all fall within the scope of the above model.

As is detailed in Section IV, the information-theoretic capacity of the channel described in (1) is given by the maximum of the mutual information, as in

$$C = \max_{Q \in \mathcal{Q}} I(\boldsymbol{x}, \boldsymbol{y}) = \max_{Q \in \mathcal{Q}} \log \det(\boldsymbol{I} + \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^*)$$

over all feasible covariance matrices $\boldsymbol{Q}$ of the input $\boldsymbol{x}$, where '$\cdot^*$' denotes the conjugate transpose.

In the following we assume complete channel state information in that $\boldsymbol{H}$ is known at the transmitter and the receiver. In this work, we focus on sets $\mathcal{Q}$ obtained by constraining the p-norm of the vector of eigenvalues $(\lambda_1(\boldsymbol{Q}), \ldots, \lambda_t(\boldsymbol{Q}))$ of the power matrices $\boldsymbol{Q} \in \mathcal{Q}$ by some constant L, i.e.,

$$\left(\sum_{i=1}^{t} \lambda_i^p(\boldsymbol{Q}) \right)^{1/p} \leq L. \tag{3}$$

In the case of sum power constraints at the transmitter, the capacity and the associated optimum power allocation strategy is given by the water-filling principle, see [1], [4], [5]. This well-known solution is obtained as the special case $p = 1$ in a rather natural way. In this correspondence, we show that the more general formulation, in terms of the p-norm constraint, enables one to approach other more complicated constraints, such as when the standard sum power constraint is accompanied by constraints on the powers of the individual components. This application, and others, are discussed further in Section II.

This correspondence provides an algorithm to solve the p-norm constraint that is almost as fast as water pouring in general, and reduces to water pouring when $p = 1$. From a mathematical point of view, the present approach is conceptually simple, providing an elegant generalization of the concept of water-filling.

From a methodological point of view we proceed as follows. First, the optimum solution is characterized by the fact that directional derivatives in any direction are nonpositive. Then, by evaluating this condition for p-norm constraints, explicit solutions are obtained, generalizing some of the topics in [6] and [7]. Another way to achieve the optimal solution is the well-known KKT-approach, as outlined in the Appendix. Since the concept based on directional derivatives is new and self-contained, this approach is explicated in the following sections.

This correspondence is structured as follows. We start with motivation for the p-norm constraint in Section II. We introduce the general concept of directional derivatives in Section III. Section IV, directional derivatives as well as gradients are determined. In Section V, the optimization problem for achieving capacity subject to eigenvalue constraints is addressed. Section VI describes an algorithm for determining the optimal solution numerically. A short summary concludes our work in Section VII.

II. Motivation

The eigenvalues of $\boldsymbol{Q}$ are intimately connected to the transmit powers, as follows. First, the sum of transmit powers (the trace of $\boldsymbol{Q}$) is equal to the sum of the eigenvalues of $\boldsymbol{Q}$. Thus, the sum power constraint is retrieved by taking $p = 1$ in (3). Water-filling provides a fast algorithm to maximize capacity subject to the sum power constraint.

The practical motivation for dealing with more general values $1 \leq p \leq \infty$ is to provide a similar fast numerical method to handle constraints in addition to (or replacing) the sum power constraint. Examples of such constraints are found in [8] and the references cited therein. For example, in the example of an antenna array, it may be required to bound the dynamic range of the power amplifier at each transmit antenna, and this can be approached by replacing $p = 1$ with $p = \infty$, see [8]. This follows since

$$(q_{11}, \ldots, q_{tt}) \prec (\lambda_1(\boldsymbol{Q}), \ldots, \lambda_t(\boldsymbol{Q}))$$

where $q_{11}, \ldots, q_{tt}$ are the individual transmit powers, and where '$\prec$' denotes majorization, see [9, p.218]. In particular,

$$\max_{1 \leq i \leq t} q_{ii} \leq \max_{1 \leq i \leq t} \lambda_i(\boldsymbol{Q})$$

which implies that bounding the maximum eigenvalue with $p = \infty$ also provides an upper bound to the maximum average antenna power.

To see why values of p within the interval $1 < p < \infty$ may be of interest, consider the case in which there is a constraint, β, on the sum of the average transmit powers, but there is also a constraint, α, on the average power of each individual component. Suppose that $\beta/t \leq \alpha \leq \beta$, which is the interesting case in which it is necessary to state both sum and individual constraints (if $\alpha \geq \beta$ then the problem reduces to water filling, and if $0 \leq \alpha \leq \beta/t$, we reduce to the $p = \infty$ case mentioned above). The following lemma then applies.

Lemma 1: Let $\beta/t \leq \alpha \leq \beta$ and

$$p = \frac{\ln t}{\ln(\alpha t/\beta)}. \tag{4}$$

Then any $\boldsymbol{Q}$ satisfying (3) with $L = \alpha$ also satisfies both the max power constraint $\lambda_{\max}(\boldsymbol{Q}) \leq \alpha$ and the sum power constraint $\sum_{i=1}^{t} \lambda_i(\boldsymbol{Q}) \leq \beta$. Moreover, the above p is the largest possible value with this property.

Proof: First, it is immediate that if (3) holds, then the maximum eigenvalue cannot exceed α, and the considerations above imply the same is true of the maximum individual power. Now consider the optimization problem

$$\max \sum_{i=1}^{t} \lambda_i \ s.t. \ \left(\sum_{i=1}^{t} \lambda_i^p\right)^{1/p} \leq \alpha. \tag{5}$$

Provided $1 < p < \infty$, this has a unique optimum value of $\alpha t^{1-1/p}$ achieved when all λ_i are equal. At $p = 1$ this is still true, but the solution is then not unique. The value of p in (4) is the unique solution to the equation $\alpha t^{1-1/p} = \beta$ and under the condition of the lemma, it also satisfies $1 \leq p < \infty$. It follows that if $\boldsymbol{Q}$ satisfies (3) with $L = \alpha$, then it is necessary that $\sum_{i=1}^{t} \lambda_i(\boldsymbol{Q}) \leq \beta$. Since the trace of a matrix is the sum of its eigenvalues it also follows that the system satisfies a sum power constraint of β. Note that if p takes a value larger than the value specified in (4) then the maximum value in (5) exceeds β, and hence one cannot guarantee that the sum power constraint of β will be satisfied. $\square$

We conclude that if the exponent p is appropriately chosen then constraint (3) is sufficient to jointly meet both a max average power constraint with threshold α and a sum average power constraint with upper bound β. This fact is illustrated in Fig. 1, which depicts the intersection of the sum power constrained region using $\beta = 1.55$ with the max power constrained region, using $\alpha = 1$ (dark shaded jointly with light area) in $\mathbb{R}^2$, i.e., the set

$$\{(\lambda_1, \lambda_2) \mid \lambda_1 + \lambda_2 \leq 1.55, 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1\}.$$

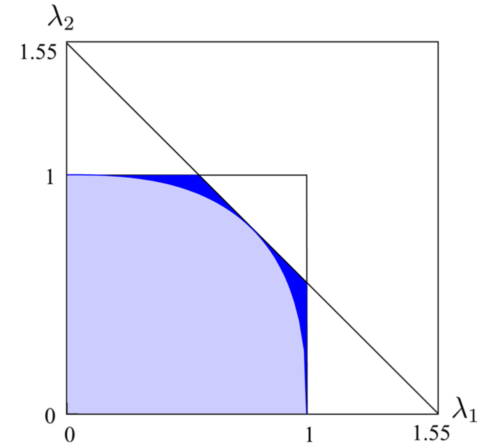

Fig. 1. The intersection of sum power constraints with $L_1 = 1.55$ and max power constraints with $L_2 = 1$ (the union of dark shaded and light shaded areas) approximated by the p-norm constraints with $L = 1$ and $p = 2.7$ (light shaded area).

This region can be approximated by the p-norm constrained region (3) with $L = \alpha = 1$ and $p = 2.7$ (light shaded area). Clearly, the constraint (3) is only sufficient, in the sense that the p-norm region is only a subset of the intersection of the regions defined by each individual constraint. However, it is a close approximation in this case.

Although it is certainly possible to formulate an optimization problem including both the sum constraint, and the individual constraints, there is no known algorithm anywhere near as fast as water pouring in general. A contribution of the present correspondence is to show that there is such an algorithm, if we replace the separate constraints by the single constraint of the form (3).

The constraint (3) may also be useful in modeling other system constraints. For example, it may be that the transmit power radiated in certain directions should be bounded (as discussed in [8]) in addition to the requirement that the sum power constraint must hold. This situation arises in multiuser applications, when the interference created by the transmitter must also be considered.

For a unit vector $\boldsymbol{u} \in \mathbb{C}^t$, the power radiated in direction $\boldsymbol{u}$ is $\boldsymbol{u}^*\boldsymbol{Q}\boldsymbol{u}$. This is the interference power experienced at another node, if the channel matrix to the other node is $\boldsymbol{G}$ (an $r' \times t$ matrix if the receiver has r' dimensions), the linear receiver vector used at that node is $\boldsymbol{c} \in \mathbb{C}^{r'}$, and the direction $\boldsymbol{u}$ is identified with the vector $\boldsymbol{G}^*\boldsymbol{c}$. If the precise directions used by other nodes are not known by the transmitter, then it may be useful to bound the power radiated in *all* possible directions. This is analogous to providing an upper bound on power spectral density (or, more generally, a spectral mask) in the frequency domain.

Using the theory of Rayleigh quotients

$$\max_{\boldsymbol{u}^*\boldsymbol{u}} \boldsymbol{u}^*\boldsymbol{Q}\boldsymbol{u} = \lambda_{\max}(\boldsymbol{Q})$$

which implies that $\lambda_{\max}(\boldsymbol{Q}) \leq \alpha$ is sufficient to ensure that $\boldsymbol{u}^*\boldsymbol{Q}\boldsymbol{u} \leq \alpha$ for any direction $\boldsymbol{u}$. Thus, even if there are no explicit individual power constraints, a bound on the maximum eigenvalue provides an effective spectral mask on the radiated power in any direction. The present correspondence provides a way to couple this constraint with a sum power constraint, to obtain a fast algorithm analogous to water filling.

III. Preliminaries

In this section, we briefly summarize the concept of directional derivatives and its relation to the optimization of concave functions. Let f be a real-valued concave function with convex domain $\mathcal{C}$ and

$\hat{x}, x \in \mathcal{C}$. The directional derivative of f at $\hat{x}$ in the direction of x is defined as

$$\begin{aligned} Df(\hat{x}, x) &= \lim_{\alpha \to 0+} \frac{1}{\alpha} \left[f\left((1-\alpha)\hat{x} + \alpha x\right) - f(\hat{x}) \right] \\ &= \frac{d}{d\alpha} f\left((1-\alpha)\hat{x} + \alpha x\right) \Big|_{\alpha = 0+} \end{aligned} \tag{6}$$

see, e.g., [10], or [11] (the latter with a slightly modified definition). Since f is concave, $\left(f((1-\alpha)\hat{x} + \alpha x) - f(\hat{x})\right)/\alpha$ is monotone increasing with decreasing $\alpha \geq 0$, and the directional derivative always exists, cp. [11, Theorem 23.1].

If $\mathcal{C}$ is a subset of a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, it is well known that

$$Df(\hat{x}, x) = \langle \nabla f(\hat{x}), x - \hat{x} \rangle,$$

whenever ∇f, the derivative of f (see [11, Sec. 25]) exists.

Optimum points are characterized by directional derivatives as follows, for a proof see [10].

Proposition 2: Let $\mathcal{C}$ be a convex set and $f : \mathcal{C} \to \mathbb{R}$ a concave function. Then the maximum of f is attained at $\hat{x}$ if and only if $Df(\hat{x}, x) \leq 0$ for all $x \in \mathcal{C}$.

IV. Directional Derivatives of Mutual Information

Following the arguments in [1], the mutual information for the linear Gaussian noise channel (1) is upper bounded by

$$I(\boldsymbol{x}, \boldsymbol{y}) = H(\boldsymbol{y}) - H(\boldsymbol{n}) \leq \log\det(\boldsymbol{I}_r + \boldsymbol{HQH}^*)$$

with equality if $\boldsymbol{x} \sim \text{SCN}(\boldsymbol{0}, \boldsymbol{Q})$. Hence, the capacity of vector channel (1) subject to mean power constraints (2) is derived as the maximum of the mutual information over all admissible input distributions of $\boldsymbol{x}$ as

$$C = \max_{\boldsymbol{Q} \in \mathcal{Q}} I(\boldsymbol{x}, \boldsymbol{y}) = \max_{\boldsymbol{Q} \in \mathcal{Q}} \log\det(\boldsymbol{I}_r + \boldsymbol{HQH}^*).$$

In the following we characterize the covariance matrix $\hat{\boldsymbol{Q}}$ which achieves capacity by using directional derivatives of the function

$$f : \mathcal{Q} \to \mathbb{R} : \boldsymbol{Q} \mapsto \log\det(\boldsymbol{I}_r + \boldsymbol{HQH}^*).$$

From Ky Fan's inequality ([12, Lemma1]) it follows immediately that f is concave whenever its domain $\mathcal{Q}$ is convex.

Proposition 3: Let $\mathcal{Q}$ be convex and $\hat{\boldsymbol{Q}}, \boldsymbol{Q} \in \mathcal{Q}$. The directional derivative of f at $\hat{\boldsymbol{Q}}$ in the direction of $\boldsymbol{Q}$ is given by

$$Df(\hat{\boldsymbol{Q}}, \boldsymbol{Q}) = \operatorname{tr}\left(\boldsymbol{H}^*(\boldsymbol{I}_r + \boldsymbol{H}\hat{\boldsymbol{Q}}\boldsymbol{H}^*)^{-1}\boldsymbol{H}(\boldsymbol{Q} - \hat{\boldsymbol{Q}})\right). \tag{7}$$

Proof: We exploit the chain rule for real valued functions $g(\boldsymbol{X})$ where the matrix $\boldsymbol{X}$ is itself a function of a scalar α,

$$\frac{dg}{d\alpha} = \operatorname{tr}\left(\frac{dg}{d\boldsymbol{X}}\left(\frac{d\boldsymbol{X}}{d\alpha}\right)^*\right).$$

Furthermore, we utilize that $\frac{d}{d\boldsymbol{X}}\det\boldsymbol{X} = (\det\boldsymbol{X})(\boldsymbol{X}^{-1})^*$, cf. [13]. Hence,

$$\begin{aligned} &\frac{d}{d\alpha} f(\hat{\boldsymbol{Q}} + \alpha(\boldsymbol{Q} - \hat{\boldsymbol{Q}})) \\ &= \frac{d}{d\alpha} \log\det\left(\boldsymbol{I}_r + \boldsymbol{H}\hat{\boldsymbol{Q}}\boldsymbol{H}^* + \alpha\boldsymbol{H}(\boldsymbol{Q} - \hat{\boldsymbol{Q}})\boldsymbol{H}^*\right) \\ &= \operatorname{tr}\Bigg((\boldsymbol{I}_r + \boldsymbol{H}\hat{\boldsymbol{Q}}\boldsymbol{H}^* + \alpha\boldsymbol{H}(\boldsymbol{Q} - \hat{\boldsymbol{Q}})\boldsymbol{H}^*)^{-1} \\ &\qquad \times \boldsymbol{H}(\boldsymbol{Q} - \hat{\boldsymbol{Q}})\boldsymbol{H}^* \Bigg). \end{aligned}$$

Setting $\alpha = 0$ and cyclically interchanging $\boldsymbol{H}^*$ in the trace yields representation (7) □

Representation (7) demonstrates that the directional derivative is linear in $\boldsymbol{Q} - \hat{\boldsymbol{Q}}$. From [11, Th.25.2] we conclude that f is differentiable at $\hat{\boldsymbol{Q}}$ in the Hilbert space of all complex $t \times t$ matrices endowed with the inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{tr}(\boldsymbol{AB}^*)$, see [14, p.286]. Furthermore, the gradient is given by

$$\nabla f(\hat{\boldsymbol{Q}}) = \boldsymbol{H}^*(\boldsymbol{I}_r + \boldsymbol{H}\hat{\boldsymbol{Q}}\boldsymbol{H}^*)^{-1}\boldsymbol{H}. \tag{8}$$

V. Capacity For p-Norm Constraints

To achieve capacity subject to the power constraint specified by the set $\mathcal{Q}$ one must maximize $f(\boldsymbol{Q})$ over the set of possible power assignments $\mathcal{Q}$. According to Proposition 2 the point $\hat{\boldsymbol{Q}}$ maximizes $f(\boldsymbol{Q})$ over some convex set $\mathcal{Q}$ if and only if $Df(\hat{\boldsymbol{Q}}, \boldsymbol{Q}) \leq 0$ for all $\boldsymbol{Q} \in \mathcal{Q}$. By (7) this leads to

$$\operatorname{tr}\left(\boldsymbol{H}^*(\boldsymbol{I}_r + \boldsymbol{H}\hat{\boldsymbol{Q}}\boldsymbol{H}^*)^{-1}\boldsymbol{HQ}\right) \leq \operatorname{tr}\left(\boldsymbol{H}^*(\boldsymbol{I}_r + \boldsymbol{H}\hat{\boldsymbol{Q}}\boldsymbol{H}^*)^{-1}\boldsymbol{H}\hat{\boldsymbol{Q}}\right) \tag{9}$$

for all $\boldsymbol{Q} \in \mathcal{Q}$. Hence, we obtain the following proposition.

Proposition 4: $\max_{\boldsymbol{Q} \in \mathcal{Q}} F(\boldsymbol{Q})$ is attained at $\hat{\boldsymbol{Q}}$ if and only if $\hat{\boldsymbol{Q}}$ is a solution of

$$\max_{\boldsymbol{Q} \in \mathcal{Q}} \operatorname{tr}\left(\nabla f(\hat{\boldsymbol{Q}})\boldsymbol{Q}\right). \tag{10}$$

Power constraints from matrix p-norms are considered in the following. For a given $1 \leq p \leq \infty$ they are defined on the set of nonnegative Hermitian $t \times t$ matrices as

$$\|\boldsymbol{A}\|_p = \left(\sum_{i=1}^{t} \lambda_i^p(\boldsymbol{A})\right)^{1/p}$$

where $\lambda_i(\boldsymbol{A})$, $i = 1, \ldots, t$, denote the eigenvalues of $\boldsymbol{A}$.

Sum power constraints are contained as the special case $p = 1$. Maximizing capacity here follows the well known water-filling principle, where the solution is obtained by water filling onto the inverse positive eigenvalues of $\boldsymbol{H}^*\boldsymbol{H}$, cf. [1]. The opposite extreme $p = \infty$ corresponds to maximum eigenvalue constraints, since $\lim_{p \to \infty} \|\boldsymbol{A}\|_p = \lambda_{\max}(\boldsymbol{A})$, the maximum eigenvalue of $\boldsymbol{A}$. The optimum solution in this case is a multiple of the identity matrix, cf. [6].

For general $1 \leq p \leq \infty$ and $L > 0$ the constraining set is given by

$$\mathcal{Q}_{p,L} = \{\boldsymbol{Q} \geq \boldsymbol{0} \mid \|\boldsymbol{Q}\|_p \leq L\}.$$

We use the notation $\boldsymbol{Q} \geq \boldsymbol{0}$ to indicate $\boldsymbol{Q}$ Hermitian nonnegative definite.

The corresponding maximum in (10) can be explicitly determined as follows.

Proposition 5: Let $p, q \geq 1$ be conjugate, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\max_{\boldsymbol{Q} \geq \boldsymbol{0}, \|\boldsymbol{Q}\|_p \leq L} \operatorname{tr}\left(\nabla f(\hat{\boldsymbol{Q}})\boldsymbol{Q}\right) = L\|\nabla f(\hat{\boldsymbol{Q}})\|_q. \tag{11}$$

To see this we exploit the fact that $\operatorname{tr}(\boldsymbol{AB}) \leq \sum \lambda_{(i)}(\boldsymbol{A})\lambda_{(i)}(\boldsymbol{B})$ for the ordered eigenvalues of the nonnegative definite Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, see [9, H.1.g, p. 248]. Together with Hölder's inequality and the fact that $\|\boldsymbol{Q}\|_p \le L$ over $\mathcal{Q}_{p,L}$ the following chain of inequalities is obtained:

$$\begin{aligned}
&\max_{\boldsymbol{Q}\in\mathcal{Q}_{p,L}} \operatorname{tr}\left(\nabla f(\hat{\boldsymbol{Q}})\boldsymbol{Q}\right) \\
&\quad \le \max_{\boldsymbol{Q}\in\mathcal{Q}_{p,L}} \sum_{i=1}^{t} \lambda_{(i)}\left(\nabla f(\hat{\boldsymbol{Q}})\right)\lambda_{(i)}(\boldsymbol{Q}) \\
&\quad \le \left(\sum_{i=1}^{t} \lambda_{(i)}^{q}\left(\nabla f(\hat{\boldsymbol{Q}})\right)\right)^{1/q} \\
&\max_{\boldsymbol{Q}\in\mathcal{Q}_{p,L}} \left(\sum_{i=1}^{t} \lambda_{(i)}^{p}(\boldsymbol{Q})\right)^{1/p} \\
&\quad \le L\|\nabla f(\hat{\boldsymbol{Q}})\|_q.
\end{aligned}$$

In the case $p,q>1$ equality holds if $\lambda_{(i)}(\boldsymbol{Q}) = \alpha\lambda_{(i)}^{q-1}\left(\nabla f(\hat{\boldsymbol{Q}})\right)$, $\boldsymbol{Q}$ has the same system of unitary eigenvectors as $\nabla f(\hat{\boldsymbol{Q}})$, and α is such that $\|\boldsymbol{Q}\|_p = L$. If $p=1$, then equality holds if $\boldsymbol{Q} = L\boldsymbol{v}\boldsymbol{v}^*$, where $\boldsymbol{v}$ denotes a normalized eigenvector corresponding to $\lambda_{\max}\left(\nabla f(\hat{\boldsymbol{Q}})\right)$. If $p=\infty$, equality holds for $\boldsymbol{Q} = L\boldsymbol{I}_t$. In summary, (11) follows.

Now, in combining Propositions 4 and 5, we get

Proposition 6: Let $p,q \ge 1$ be conjugate. Capacity, i.e., $\max_{\boldsymbol{Q}\in\mathcal{Q}_{p,L}} f(\boldsymbol{Q})$ is attained at power distribution $\hat{\boldsymbol{Q}} \in \mathcal{Q}_{p,L}$ if and only if

$$L\|\nabla f(\hat{\boldsymbol{Q}})\|_q = \operatorname{tr}\left(\nabla f(\hat{\boldsymbol{Q}})\hat{\boldsymbol{Q}}\right). \tag{12}$$

Once we can solve the above equation for $\hat{\boldsymbol{Q}}$, an optimum power allocation is found. For this purpose let

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Gamma}^{1/2}\boldsymbol{V}^*$$

denote the singular value decomposition of the channel matrix $\boldsymbol{H}$ with unitary $(r\times r)$ matrix $\boldsymbol{U}$, unitary $(t\times t)$ matrix $\boldsymbol{V}$, and $(r\times t)$ matrix $\boldsymbol{\Gamma}^{1/2}$ containing the diagonal matrix of singular values in the upper left corner and zeros elsewhere. Let γ_i denote the identical positive eigenvalues of $\boldsymbol{H}\boldsymbol{H}^*$ and $\boldsymbol{H}^*\boldsymbol{H}$, augmented by zeros whenever appropriate.

In the following, we try to find a solution of (12) in the class of power allocations:

$$\hat{\boldsymbol{Q}} = \boldsymbol{V}\operatorname{diag}(\hat{q}_1,\ldots,\hat{q}_t)\boldsymbol{V}^*, \quad \hat{q}_i \ge 0, \quad \left(\sum_i \hat{q}_i^p\right)^{1/p} \le L.$$

The first step is to evaluate (12) for $\hat{\boldsymbol{Q}}$ of the above type. It is easy to see that the following representations hold:

$$L\|\nabla f(\hat{\boldsymbol{Q}})\|_q = L\left(\sum_{i=1}^{t}\left(\frac{\gamma_i}{1+\gamma_i\hat{q}_i}\right)^q\right)^{1/q} \tag{13}$$

$$\operatorname{tr}\left(\nabla f(\hat{\boldsymbol{Q}})\hat{\boldsymbol{Q}}\right) = \sum_{i=1}^{t}\left(\frac{\gamma_i\hat{q}_i}{1+\gamma_i\hat{q}_i}\right). \tag{14}$$

We first single out the case $p=\infty$ with $\|\hat{\boldsymbol{Q}}\|_\infty = \max_i \hat{q}_i$. Then, equality of (13) and (14) holds if $\hat{q}_i = L$ for all $i=1,\ldots,t$ with $\gamma_i > 0$, and $\hat{q}_i = 0$, otherwise. Note that for $\gamma_i = 0$ any other $q_i \in [0,L]$ ensures equality and yields an admissible solution as well.

In the case $p=1$ let $\hat{q}_i = (\nu - 1/\gamma_i)^+$, ν such that $\sum_{i=1}^t \hat{q}_i = L$. Some algebra shows that in this case (13) and (14) have the same value L/ν and hence are equal.

For general $1<p<\infty$, let $\hat{q}_i \ge 0$ be such that

$$\frac{\nu\gamma_i}{1+\gamma_i\hat{q}_i} = \hat{q}_i^{p-1} \tag{15}$$

for all $i=1,\ldots,t$, for some $\nu>0$ satisfying

$$\left(\sum_{i=1}^{t}\hat{q}_i^p\right)^{1/p} = L. \tag{16}$$

By Hölder's inequality

$$\sum_{i=1}^{t}\left(\frac{\gamma_i\hat{q}_i}{1+\gamma_i\hat{q}_i}\right) \le \left(\sum_{i=1}^{t}\hat{q}_i^p\right)^{1/p}\left(\sum_{i=1}^{t}\left(\frac{\gamma_i}{1+\gamma_i\hat{q}_i}\right)^q\right)^{1/q}.$$

Applying (16), we obtain that (13) equals (14).

For positive γ_i (15) can equivalently be written as $\hat{q}_i^p + \frac{1}{\gamma_i}\hat{q}_i^{p-1} = \nu$. In summary, we have proven the following central result.

Theorem 7: For $1<p<\infty$ let $\hat{q}_i \ge 0$, $i=1,\ldots,t$, denote the unique solution of the system of equations

$$\begin{aligned}
&\hat{q}_i = 0, \text{ if } \gamma_i = 0 \\
&\hat{q}_i^p + \frac{1}{\gamma_i}\hat{q}_i^{p-1} = \nu, \text{ if } \gamma_i > 0, \ \nu \text{ such that } \left(\sum_{i=1}^{t}\hat{q}_i^p\right)^{1/p} = L.
\end{aligned} \tag{17}$$

For the limiting case $p=1$, it holds that

$$\hat{q}_i = \left(\nu - \frac{1}{\gamma_i}\right)^+, \ \nu \text{ such that } \sum_{i=1}^{t}\hat{q}_i = L. \tag{18}$$

In the case $p=\infty$ let $\hat{q}_i = L$ for all $i=1,\ldots,t$ with $\gamma_i>0$, and $\hat{q}_i = 0$, otherwise.

Then, for any $1\le p\le\infty$

$$\hat{\boldsymbol{Q}} = \boldsymbol{V}\operatorname{diag}(\hat{q}_1,\ldots,\hat{q}_t)\boldsymbol{V}^*$$

is a solution of

$$\max_{\boldsymbol{Q}\ge\boldsymbol{0},\|\boldsymbol{Q}\|_p\le L} \log\det(\boldsymbol{I}+\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^*)$$

and hence represents an optimal power assignment.

The function $q_i^p + \frac{1}{\gamma_i}q_i^{p-1}$ is monotone in q_i for any $p>1$; thus, a solution of (17) always exists for any $L>0$. Observe that except for the case $p=1$ all positive eigenvalues γ_i receive a positive amount of power.

A graphical solution of Proposition 7 is represented in Fig. 2. The solid, dotted, and dashed lines correspond to values $p=2$ and $\gamma_1=4$, $\gamma_2=3$, $\gamma_3=2$, and ν is set to 0.4. The optimum arguments can be read off from the x-axis as 0.52, 0.48, and 0.43, respectively.

The well known water-filling solution (18) is nicely obtained as a special limiting case of (17) in Theorem 7. This fact is illustrated by the curves plotted in Fig. 3. The curves are analogous to those in Fig. 2, but the p-values are now set to $p = 2.5, 1.33, 1.000001$ (from right to left). For $p=1$ the corresponding $\hat{q}$-values are $0.00, 0.067, 0.15$, respectively, exactly those obtainable from classical water-filling.

VI. A Numerical Algorithm

We provide a fast quadratically convergent algorithm for finding numerical solutions to the system of (17). It consists of nested Newton iterations of the general type $x_{n+1} = x_n - f(x_n)/f'(x_n)$, useful for finding a solution x of $f(x)=0$. On the one hand, the inverses $g_i^{-1}(\nu)$ of the monotone functions

$$g_i(x) = x^p + \frac{1}{\gamma_i}x^{p-1}, \quad x\ge 0$$

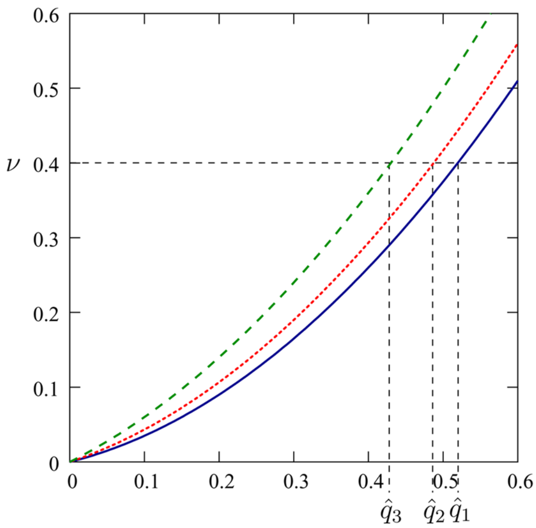

Fig. 2. The curves $\hat{q}_i^p + \hat{q}_i^{p-1}/\gamma_i$ for $p = 2, \gamma_1 = 4$ (solid), $\gamma_2 = 3$ (dotted), $\gamma_3 = 2$ (dashed). $\nu = 0.4$ corresponds to the optimum power assignments indicated by $\hat{q}_i$ on the x-axis.

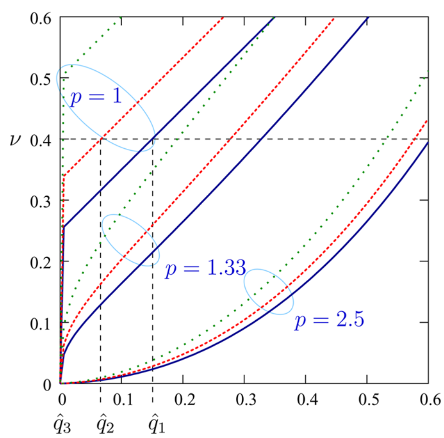

Fig. 3. Visualizing the limiting case $\hat{q}_i^p + \hat{q}_i^{p-1}/\gamma_i$ as $p \to 1$. Further values are $\gamma_1 = 4$ (solid), $\gamma_2 = 3$ (dotted), $\gamma_3 = 2$ (dashed). $\nu = 4$ leads to the optimum water-filling solution $\hat{q}_i$ indicated on the x-axis.

are determined, on the other hand a zero of the monotone function

$$H(\nu) = \sum_i \left[g_i^{-1}(\nu)\right]^p - L^p, \quad \nu \geq 0$$

is found, where the summation is over all indices i with $\gamma_i > 0$. The first task is solved for $\nu = \nu^{(n)}$ by the iteration

$$x_{i,k+1}^{(n)} = x_{i,k}^{(n)} - \frac{\left(x_{i,k}^{(n)}\right)^p + \left(x_{i,k}^{(n)}\right)^{p-1}/\gamma_i - \nu^{(n)}}{p\left(x_{i,k}^{(n)}\right)^{p-1} - (p-1)\left(x_{i,k}^{(n)}\right)^{p-2}/\gamma_i}$$
$$k \in \mathbb{N}_0 \tag{19}$$

for indices i with positive γ_i.

The outer iteration uses

$$H'(\nu) = \sum_i p\left[g_i^{-1}(\nu)\right]^{p-1} g_i^{-1\prime}(\nu) = \sum_i \frac{1}{1 + \frac{p-1}{p\gamma_i}\left[g_i^{-1}(\nu)\right]^{-1}}$$

to update $\nu^{(n)}$ as

$$\nu^{(n+1)} = \nu^{(n)} - \frac{H(\nu^{(n)})}{H'(\nu^{(n)})}, \quad n \in \mathbb{N}_0.$$

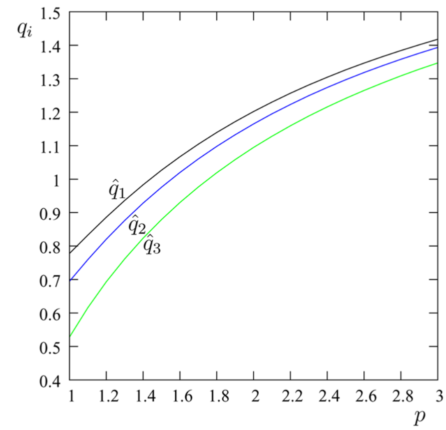

Fig. 4. Optimum values $\hat{q}_1, \hat{q}_2, \hat{q}_3$ as a function of $p \in [1, 3]$ for parameters $L = 2$ and $\gamma_1 = 4, \gamma_2 = 3, \gamma_3 = 2$.

As an initial value in the iteration (19) we choose $x_{i,0}^{(n)} = g_i^{-1}(\nu_{n-1})$ from the previous step, which yields excellent convergence results (eight digits accuracy after four to six iterations), provided the outer initial value ν_0 and the initial values for computing $g_i^{-1}(\nu^{(0)})$ are appropriately chosen. Global convergence can be achieved by using the Newton–Dogleg Method instead of Newton's Method, see [15].

We have used the above algorithm to compute the optimum values $\hat{q}_1, \hat{q}_2, \hat{q}_3$ as a function of parameter $p \in [1, 3]$ for $L = 2$ and channel eigenvalues $\gamma_1 = 4, \gamma_2 = 3, \gamma_3 = 2$. The resulting curves are plotted in Fig. 4. There is numerical evidence that each tends monotonically to L (in this case, $L = 2$) as $p \to \infty$. From this evidence, it seems that asymptotically $|\hat{q}_i - \hat{q}_j|$ tends to zero. However, in cases $p = 1$ and small values of L some of the $\hat{q}_i$ may be identically zero as is known from sum power constrained water-filling. A strict demonstration of the apparent fact that the solutions are less spread out for increasing p is still an open question, although it certainly accords with the known solution when $p = \infty$.

VII. Conclusion

The central topic of the present correspondence is the derivation of the capacity of Gaussian vector channels over a general constraining class of power assignments. Constraints are expressed by bounding the p-norm of the vector of eigenvalues of the power matrix. Directional derivatives are used to identify optimal solutions. The well known water-filling principle turns out as a special limiting case when $p = 1$. The investigations for general $p > 1$ allow for the approximate handling of the case when the standard sum power constraint is accompanied by power constraints on the individual vector components. We also provide a fast quadratically convergent algorithm for determining numerical solutions. There is numerical evidence that for increasing p the optimum solutions are monotonically increasing and less spread out.

Appendix

A direct proof of the main result by using KKT theory was suggested by a reviewer. We give a brief outline of this alternative method here. The problem

$$\max \log\det(\boldsymbol{I} + \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^*) \text{ such that } \boldsymbol{Q} \geq \boldsymbol{0}, \|\boldsymbol{Q}\|_p \leq L$$

is a convex optimization problem since the objective function is concave and the constraint set is convex (see Section IV). For reasons of differentiability we exclude the easy case $p = \infty$, hence assuming $1 \leq p < \infty$ in the following.

The Hermitian matrices $\boldsymbol{Q}$ and $\boldsymbol{U}^*\boldsymbol{Q}\boldsymbol{U}$ have the same eigenvalues for any unitary matrix $\boldsymbol{U}$. Hence, following the arguments in [1] the problem may be equivalently reduced to

$$\max \sum_{i=1}^{t} \log(1+\gamma_i q_i) \text{ such that } q_k \geq 0, \sum_{i=1}^{t} q_i^p \leq L^p$$

where $\gamma_i \geq 0$, $i = 1,\dots,t$, denote the eigenvalues of $\boldsymbol{H}^*\boldsymbol{H}$, corresponding to the spectral decomposition $\boldsymbol{H}^*\boldsymbol{H} = \boldsymbol{V}\operatorname{diag}(\gamma_1,\dots,\gamma_t)\boldsymbol{V}^*$ for some unitary matrix $\boldsymbol{V}$. The optimal solution $\hat{\boldsymbol{Q}}$ to the original problem is obtained as $\hat{\boldsymbol{Q}} = \boldsymbol{V}\operatorname{diag}(\hat{q}_1,\dots,\hat{q}_t)\boldsymbol{V}^*$ from the optimal solution $(\hat{q}_1,\dots,\hat{q}_t)$ of the reduced problem.

Again, the reduced problem is a convex optimization problem which can be solved via the KKT optimality conditions, see [16]. The Lagrangian is given by

$$L(\boldsymbol{q},\boldsymbol{\eta},\zeta) = \sum_{i=1}^{t}\log(1+\gamma_i q_i) + \sum_{i=1}^{t} q_i\eta_i + \zeta\left(L^p - \sum_{i=1}^{t} q_i^p\right)$$

with the notation $\boldsymbol{q} = (q_1,\dots,q_t)$ and $\boldsymbol{\eta} = (\eta_1,\dots,\eta_t)$. The optimality conditions are (cf. [16, Ch. 5.5.3])

$$\begin{aligned}\frac{\partial L(\boldsymbol{q},\boldsymbol{\eta},\zeta)}{\partial q_k} &= \frac{\gamma_k}{1+\gamma_k q_k} + \eta_k - \zeta p q_k^{p-1} = 0\\ q_k,\eta_k &\geq 0, \quad k = 1,\dots,t\\ \eta_k q_k &= 0, \quad k=1,\dots,t\\ L^p - \sum_{i=1}^{t} q_i^p &= 0.\end{aligned}$$

If $\gamma_k > 0$ we can solve these equations through the following chain of transformations.

$$\begin{aligned}\frac{\gamma_k}{1+\gamma_k q_k} &= \zeta p q_k^{p-1} - \eta_k\\ \frac{\gamma_k q_k^{1-p}}{1+\gamma_k q_k} &= \zeta p - \eta_k q_k^{1-p}\\ \frac{1+\gamma_k q_k}{\gamma_k} q_k^{p-1} &= \frac{1}{\zeta p}\\ q_k^p + \frac{1}{\gamma_k} q_k^{p-1} &= \nu\end{aligned}$$

with $\nu = 1/(\zeta p)$. If $\gamma_k = 0$ we set $q_k = 0$, and select $\nu = 1/(\zeta p)$ such that $\sum_{i=1}^{t} q_i^p = L^p$. We have thus obtained a solution to the KKT optimality conditions, which corresponds exactly to conditions (17) in Theorem 7.

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