Visualization and Analysis of the OFDMA Downlink Capacity Region

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***Abstract*— In this paper we introduce a symmetry-based approach to visualize the OFDMA downlink capacity region and its convex hull. These sets are compared to numerically analyse the fact that the capacity region of an OFDMA system with an infinite number of subcarriers is convex. We believe that this approach offers a unique perspective on the structural properties of the capacity region which in turn might spawn new ideas concerning the complex problem of allocating subcarriers in order to maximize weighted sum rates (WSRmax problem).**

***Index Terms*— OFDMA, capacity region, convex hull, WSRmax**

I. Introduction

Resource allocation in OFDMA systems is a complex and computationally prohibitive task. Single points on the boundary of the OFDMA downlink capacity region can be obtained by applying convex optimization techniques to varying weighted sum rate maximization problems. As the capacity region is in general non-convex, this Lagrange Duality approach relies on the results of [1], ensuring a vanishing duality gap as the number of subcarriers approaches infinity.

An important open problem in network information theory is the derivation of the capacity region of a multiple access channel. Structural results of the capacity region are of great importance as, e.g., convexity would allow for a toolset of efficient and reliable algorithms in resource allocation. This correspondence investigates the capacity region of an OFDMA system by using a symmetry-based approach.

Section II describes the system model and formulates the problem of resource allocation and rate maximization in an OFDMA system. Section III covers the convex hull of the capacity region and its importance to the weighted sum rate maximization problem. In Section IV the symmetry-based approach is explained in detail, and Section V covers the visualization and numerical results. Section VI concludes the paper.

II. System Model and Problem Formulation

We assume a basic OFDMA downlink model with U users, N subcarriers equally dividing the bandwith, a fixed power constraint P_{tot} and perfect channel state information. For user u on subcarrier n, the channel gain is denoted by $g_{u,n}$. The channel is completely characterized by the channel gain to noise ratio (CNR) matrix c with

$$c_{u,n} = \frac{|g_{u,n}|^2}{\sigma_{u,n}^2},$$

where $\sigma_{u,n}^2$ is the variance of the zero-mean independent and identically distributed Gaussian noise that is added at the receiver part.

Let $p_{u,n}$ denote the power of user u on subcarrier n. Then, the achievable rate $r_{u,n}$ computes to

$$r_{u,n} = \frac{1}{2}\log_2(1 + p_{u,n} \cdot c_{u,n}) \text{ bits/dim.}$$

Given a weight vector $w = (w_1, \dots, w_U) \geq 0$ we consider the weighted sum rate maximization problem

$$\text{maximize} \quad \sum_{u=1}^{U} w_u \sum_{n=1}^{N} r_{u,n} \tag{1}$$

$$\text{such that} \quad \sum_{u=1}^{U}\sum_{n=1}^{N} p_{u,n} \leq P_{tot}, \tag{2}$$

$$p_{u,n} \geq 0 \quad \forall\, u \,\forall\, n, \tag{3}$$

$$\forall\, n\, \exists\, u \text{ s.t. } p_{k,n} = 0 \quad \forall\, k \neq u. \tag{4}$$

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Following [2], this optimization problem will subsequently be referred to as the WSRmax problem. In (2), "$\leq$" can be replaced by "=", as using more power will always result in higher rates. Note that (4) requires that each subcarrier is used by at most one user. A solution to the WSRmax problem is a power allocation matrix $p = (p_{u,n})$ that satisfies the constraints such that there is no other viable power allocation with a greater weighted sum rate. In a slight abuse of notation, the corresponding sum rate vector is also called a solution.

The complexity of the WSRmax problem lies within the subcarrier allocation. For any given subcarrier assignment, there is a multi-user waterfilling solution that maximizes the weighted sum rate. However, there are U^N distinct allocations, which makes any kind of exhaustive search unfeasible.

III. Capacity Region and Convex Hull

Let $R_u = \sum_{n=1}^{N} r_{u,n}$, $R = (R_1, \ldots, R_U)$. The capacity region of the OFDMA system described above is defined as

$$\mathcal{C} = \mathcal{C}(P_{tot}, c) = \left\{ R \in \mathbb{R}^U_{\geq 0} \;\middle|\; \begin{array}{c} \exists\, p_{u,n} \text{ such that} \\ (2), (3), (4) \text{ hold} \end{array} \right\}.$$

The set $\mathcal{C}$ is in general not convex. Denote by $\mathcal{H} = \mathcal{H}(\mathcal{C})$ the convex hull of $\mathcal{C}$, and by $\Delta\mathcal{H}$ the set of pareto optimal points of $\mathcal{H}$. This is a subset of the boundary $\partial\mathcal{H}$ of $\mathcal{H}$, namely

$$\Delta\mathcal{H} = \overline{\{R \in \partial\mathcal{H} \,|\, R > 0\}},$$

where $\overline{\mathcal{M}}$ is defined as the closure of a set $\mathcal{M}$. With these definitions, the following holds:

$$\mathcal{C} \cap \Delta\mathcal{H} = \{R \,|\, \exists\, w \geq 0 \text{ s.t. } R \text{ solves WSRmax}\}.$$

Therefore the solution(s) to every single WSRmax problem (based on the weight vector w) is located on the convex hull of the capacity region. This justifies the interest in the convex hull of the capacity region in a more natural way than the Lagrange Duality approach of convex optimization. In addition, it shows that exactly those subcarrier allocations are feasible that have a nonempty intersection with $\Delta\mathcal{H}$. We believe that this approach could be utilized to drastically reduce the number of potential subcarrier allocations, and therefore the computational complexity of the WSRmax problem. See Figure 1 for an exemplary illustration of the allocations that make up the capacity region and its convex hull. Figure 1 is discussed in detail in Sections IV and V.

On a related note, the Lagrange Duality approach relies on a vanishing duality gap as N approaches infinity, which is guaranteed if the capacity region approaches convexity with growing N. This makes it even more interesting to compare $\mathcal{C}$ and $\mathcal{H}$ for a varying number of subcarriers.

IV. A Symmetry-Based Approach

To visualize and compute the capacity region and its convex hull, several problems have to be addressed. The first is the problem of comparability, as different channel state information matrices c for a varying number of subcarriers N can greatly affect the shape and volume of the capacity region and its convex hull, which would make any kind of comparison, i.e., based on volume ratios, questionable at least. The second is the problem of complexity, as growing N makes it impossible to check every single subcarrier allocation. In this paper, we pursue a simple, yet effective, symmetry-based approach to solve both problems at once. For complexity and visualization purposes the number of users U is set to 2. However, the results can be generalized.

How does this approach work? To begin with, one needs a rather small N, i.e., $N \leq 4$, and a CNR matrix $c \in \mathbb{R}^{2\times N}$. Combined with the total power constraint P_{tot}, one gets a problem which is easy to compute. The idea now is to take another matrix $\tilde{c} = [c|\ldots|c] \in \mathbb{R}^{2\times kN}$ which consists of k copies of c. This matrix can be interpreted as dividing each of the original subcarriers into k parts, each with the same CNR. Because of this, there is almost no diversity gain. This is comparable to applying FDMA techniques to a flat channel or TDMA to a channel which is constant over time.

As the 2^N allocations which are based on the original CNR matrix c are a subset of the 2^{kN} allocations based on $\tilde{c}$, it follows that the capacity region of c is a subset of the capacity region of $\tilde{c}$, which extends to the convex hulls:

$$\mathcal{H}(\mathcal{C}(P_{tot}, c)) \subseteq \mathcal{H}(\mathcal{C}(P_{tot}, \tilde{c})).$$

Furthermore, because of the missing diversity gain, the convex hull increases only marginally with growing k. In the simulations we noted an increase in area of less than 1% when comparing $\mathcal{H}(\mathcal{C}(c))$ to $\mathcal{H}(\mathcal{C}(\tilde{c}))$ for large values of k. This is a negligible difference for the purpose of comparing capacity regions with similar convex hulls, and it is fair to say that the convex hulls are basically equal. Figure 1 illustrates this point. This leads to the desired comparability of area ratios for growing k, which forms the basis of our computations.

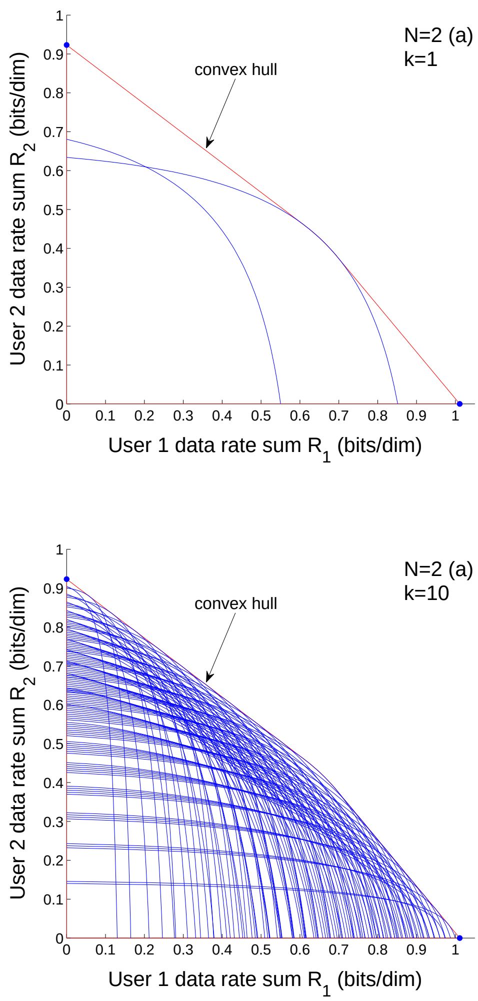

Fig. 1. The capacity regions $\mathcal{C}(c)$ and $\mathcal{C}(\tilde{c})$ (for $k = 10$).

To address the second problem, not all of the 2^{kN} subcarrier assignments are distinguishable. Applying simple combinatorics to the symmetric matrix $\tilde{c}$ shows that there are at most $(k + 1)^N$ distinct allocations. This solves the complexity problem for a wide range of values of N and k.

It should be noted that channel matrices like $\tilde{c}$ are of course most artificial. However, while in a way excluding diversity gain, the effects of the FDMA-like subcarrier sharing simulate the way that additional subcarriers lead to a more convex capacity region.

V. Visualization and Numerical Results

Figure 1 provides a good visualization of the above concepts. Note that the second plot is the result of an exhaustive search with $2 \cdot 10 = 20$ subcarriers. Instead of $2^{20} \approx 10^6$ allocations, only 121 had

TABLE I
Approaching convexity: Area ratio of capacity region $\mathcal{C}$ and convex hull $\mathcal{H}$ w.r.t. N and k.

N	CNR matrix c	$\frac{\lvert\mathcal{C}\rvert}{\lvert\mathcal{H}\rvert} \geq$	95%	99%	99.5%
1 (a)	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$k \geq$	5	11	15
1 (b)	$\begin{bmatrix} 1 \\ 5 \end{bmatrix}$	$k \geq$	6	13	19
2 (a)	$\begin{bmatrix} 2.4 & 0.9 \\ 1.4 & 1.2 \end{bmatrix}$	$2k \geq$	4	10	14
2 (b)	$\begin{bmatrix} 0.6 & 1.8 \\ 2.2 & 0.9 \end{bmatrix}$	$2k \geq$	4	6	10
3	$\begin{bmatrix} 2.4 & 1.2 & 1.5 \\ 1.4 & 0.9 & 1.8 \end{bmatrix}$	$3k \geq$	6	12	15
4	$\begin{bmatrix} 0.5 & 2.7 & 1.6 & 4.0 \\ 1.0 & 2.0 & 2.9 & 2.2 \end{bmatrix}$	$4k \geq$	4	12	16

to be considered due to the underlying symmetry.

The capacity regions were computed for different N, c, and k. Table I showcases the data used in the computations (with P_{tot} set to 2). In addition, the last columns show which number of subcarriers ($k \cdot N$) was needed to reach a certain area ratio. Note that in every computation, less than 20 subcarriers were needed for the capacity region to cover 99.5% of its convex hull, independent of the value of N. Refer to Figure 2 for a plotted version of the results.

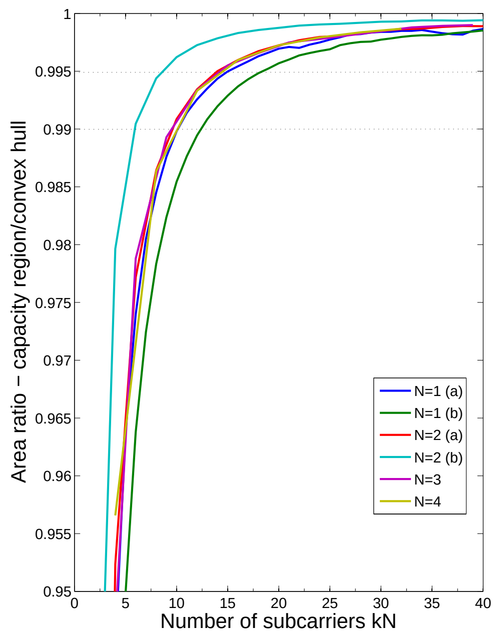

Fig. 2. Area ratio versus number of subcarriers (see Table I).

Interestingly, although naturally varying in the area ratios for $k = 1$ (no symmetry), the plots show a very similar behaviour. Even more, the amount of symmetry introduced (which decreases with N)

does not have a notable impact. In our computations, the difference in CNR played a much larger role, as can be witnessed by the plots for $N = 1$ (b) and $N = 2$ (b), respectively. However, even these adhere remarkably well to the overall pattern.

VI. Conclusion and Future Work

We have introduced a symmetry-based approach to visualize capacity regions and their convex hulls which would otherwise have been almost impossible to compute. Figure 2 as well as Table I indicate that the introduced symmetry does not have a notable effect on area ratio increase. This leads to the conclusion that regular, nonsymmetric channel matrices must adhere to similar patterns, which means capacity regions that are almost equal to their convex hulls, even for a small number of subcarriers. In fact, the number of subcarriers that was needed to obtain 99.5% of the convex hull area is much smaller than what we see in real-world applications.

The second notable observation is the impact of CNR ratios. As can be seen in Figure 1, the two straight linesegments that are part of the convex hull are basically respected even after introducing a large amount of symmetry. This suggests that the two CNR ratios (in this particular case, $2.4/1.4$ and $0.9/1.2$) play a more important role than previously thought, possibly even more important than the actual CNR values. It will be very interesting to further analyze the impact of varying CNR ratios with the goal to further improve subcarrier allocation strategies and algorithms.

One possible restriction of allocations based on the analysis of CNR ratios is shown in Figure 3. This might be a first step to reduce the computational complexity of the weighted sum rate maximization problem. We believe that a combination of convex optimization techniques and exploiting some more structural properties of the problem might lead to greater insight into the subcarrier allocation problem, even for a large number of users.

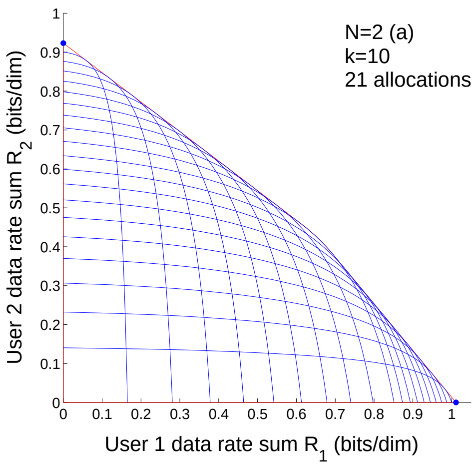

Fig. 3. Figure 1 restricted to 21 allocations.

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