Recap.

Thm 2.6. Given $M \in \mathbb{R}^{n \times n}$ symmetric with spectral decomposition $M = V \operatorname{diag}(\lambda_1, \dots, \lambda_n) V^T$, $\lambda_1 \geq \dots \geq \lambda_n$. Then

$$\min_{\substack{A \succeq 0 \\ rk(A) \leq k}} \| M - A \|_F^2$$

the minimum is attained at $A^* = V \operatorname{diag}(\lambda_1^+, \dots, \lambda_k^+, 0, \dots, 0) V^T$. $\lambda^+ = \max\{0, \lambda\}$ with the optimum value

$$\sum_{i=1}^{k} (\lambda_i - \lambda_i^+)^2 + \sum_{i=k+1}^{n} \lambda_i^2 .$$

* * *

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad \boxed{A \succeq 0}$$

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Cand. 8 $V \succeq W$ iff $V - W$ is n.n.d.

Def. 2.7 (Löwner semi-ordering) Given V and W, $V, W \succeq 0$ n.n.d. Define $V \preceq W$ if $W - V \succeq 0$ (n.n.d.)

"$\preceq$" v.s. "$\leq$"

Remark: The relation defined by "$\preceq$" is a semi-ordering, i.e.,

(i) reflexive $V \preceq V$

(ii) (anti-symmetric) if $V \preceq W$ and $W \preceq V$ then $W = V$

(iii) (transitive) If $V \preceq W$ and $W \preceq U$ then $V \preceq U$.

(Exercise)

Thm. 2.8. Given V, W n.n.d. $V = (v_{ij})$, $W = (w_{ij})$ $V \preceq W$.

$\lambda_1(V) \geq \cdots \geq \lambda_n(V)$, $\lambda_1(W) \geq \cdots \geq \lambda_n(W)$

a) $\lambda_i(V) \leq \lambda_i(W)$ $i = 1, \ldots, n$

b) $v_{ii} \leq w_{ii}$ $i = 1, \ldots, n$

c) $v_{ii} + v_{jj} - 2v_{ij} \leq w_{ii} + w_{jj} - 2w_{ij}$

d) $tr(V) \leq tr(W)$

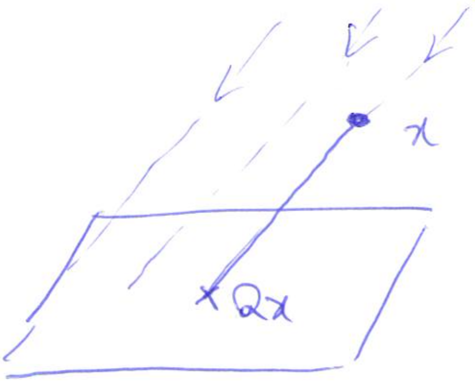
e) $det(V) \leq det(W)$ (Proof. Exercise)

Projection and Isometry.

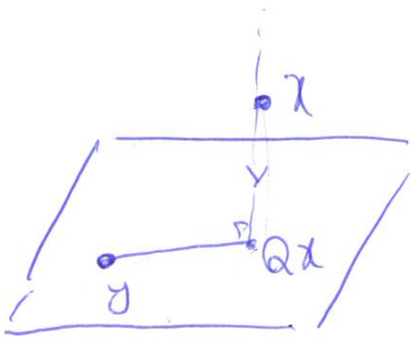
Def 2.9. $Q \in \mathbb{R}^{n \times n}$ is called a projection (matrix)

or idempotent if $Q^2 = Q$.

It is called orthogonal projection if $Q^T = Q$.



$$Q(Qx) = Qx \Rightarrow Q^2 x = Qx$$



$$x - Qx \perp \operatorname{Im}(Q)$$

$y \in \operatorname{Im}(Q)$, $\exists z \in \mathbb{R}^n$ $y = Qz$.

$$\langle y, x - Qx \rangle = y^T (x - Qx) = (Qz)^T (x - Qx)$$

$$= z^T \underline{\underline{Q}}^T (x - Qx)$$

$$= z^T Q (x - Qx) = z^T (Qx - Q^2 x) = 0.$$

Lemma 2.10. $M = V \Lambda V^T$, $M \in \mathbb{R}^{n \times n}_{sym}$

Then $Q = \sum_{i=1}^{k} v_i v_i^T$ is an orthogonal projection

onto $\operatorname{Im}(Q) = \langle v_1, \dots, v_k \rangle$.

Proof. $M = V \Lambda V^T$

$$Q = \sum_{i=1}^{K} v_i v_i^T$$

1) $Q^2 = Q$

$$\left(\sum_{i=1}^{K} v_i v_i^T\right)\left(\sum_{j=1}^{K} v_j v_j^T\right) = \sum_{i,j=1}^{K} v_i \underline{v_i^T v_j} v_j^T$$

$$v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \Rightarrow v_i^T v_i = \|v_i\|^2 = 1 \end{cases}$$

$$\Rightarrow Q^2 = \sum_{i=1}^{K} v_i v_i^T = Q.$$

2) $Q^T = Q$:

$$Q^T = \left(\sum_{i=1}^{K} v_i v_i^T\right)^T = \sum_{i=1}^{K} (v_i v_i^T)^T$$

$$= \sum_{i=1}^{K} (v_i^T)^T v_i^T$$

$$= \sum_{i=1}^{K} v_i v_i^T = Q$$

$\boxed{Q \text{ n.n.d ??}}$

$$\mathrm{Im}(Q) = \langle v_1, \dots, v_K \rangle$$

$x \in \mathbb{R}^n$

$$Qx = \left(\sum_{i=1}^{K} v_i v_i^T\right) x = \sum_{i=1}^{K} v_i \underbrace{v_i^T x}_{x_i \in \mathbb{R}}$$

$$= \sum_{i=1}^{K} x_i v_i$$

$$\Rightarrow \mathrm{Im}(Q) = \langle v_1, \dots, v_K \rangle$$

span of $v_1, \dots, v_K$

∘ Let Q be an orthogonal projection on $\text{Im}(Q)$

Then $I-Q$ is an orthogonal projection onto $\text{Ker}(Q)$.

1) $(I-Q)^2 = I-Q$

$$(I-Q)(I-Q) = I.I + Q.Q - I.Q - Q.I$$

$$= I + Q^2 - Q - Q = I + Q - Q - Q = I - Q$$

2) $(I-Q)^T = I-Q$ $\quad (I-Q)^T = I^T - Q^T = I - Q$.

$\text{Im}(I-Q) = ?$

$$x \in \text{Ker}(Q) \Rightarrow (I-Q)x = x - Qx = x$$

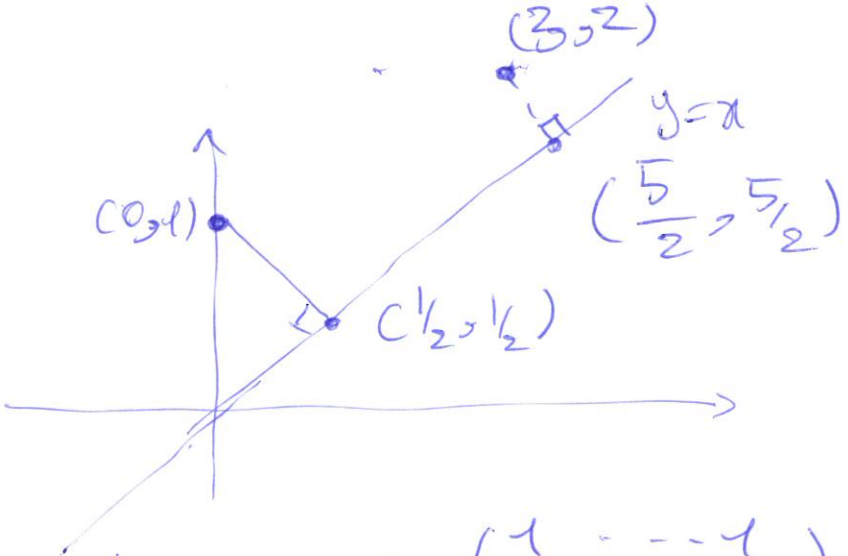
$$\Rightarrow \text{Ker}(Q) \subseteq \text{Im}(I-Q) \quad 1$$

$$x \in \text{Im}(I-Q) \Rightarrow \exists z \in \mathbb{R}^n \quad (I-Q)z = x$$

$$Qx = Q(I-Q)z = (Q - Q^2)z = 0 \Rightarrow x \in \text{Ker}(Q)$$

$$\Rightarrow \text{Im}(I-Q) \subseteq \text{Ker}(Q) \quad 2$$

$$1, 2 \Rightarrow \text{Im}(I-Q) = \text{Ker}(Q).$$



$$\frac{1}{n}\mathbb{1}_n = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

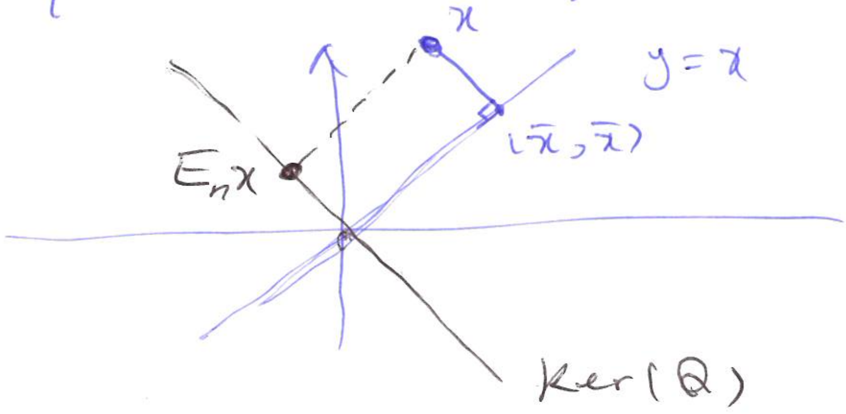
$$\frac{1}{n}\mathbb{1}_n x = \begin{bmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{bmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Centering matrix $E_n = I - \frac{1}{n}\mathbb{1}_n = \begin{pmatrix} \frac{n-1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \vdots & \ddots & & \\ -\frac{1}{n} & & \ddots & \frac{n-1}{n} \end{pmatrix}$

$$E_n x = x - \begin{pmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix} = \begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}$$

E_n is an orthogonal projection onto $\mathbb{1}_n^{\perp}$.

$$\mathbb{1}_n^{\perp} = \{x \in \mathbb{R}^n : \mathbb{1}_n^{T} x = 0\}.$$



$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \longrightarrow A^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

$$AA^{-1} = \frac{1}{ac-b^2} \begin{pmatrix} ac-b^2 & 0 \\ 0 & ac-b^2 \end{pmatrix} = I.$$

Thm 2.44. (Inverse and det. of pratitioned matrices)

Let $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ be symmetric, A inventible.

a) $$M^{-1} = \begin{pmatrix} A^{-1} + FE^{-1}F^T & -FE^{-1} \\ -E^{-1}F^T & E^{-1} \end{pmatrix}$$

$$F = A^{-1}B \qquad E = \underbrace{C - B^TA^{-1}B}_{\text{Schur complement of } A}.$$

b) $\det(M) = \det(A) \det(C - B^TA^{-1}B).$

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} A^{-1} + X & - \\ Y & - \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$A(A^{-1} + X) + BY = I \Rightarrow AX + BY = 0$$

$$B^T(A^{-1} + X) + CY = 0 \Rightarrow B^TX + CY = -B^TA^{-1}$$

$$-B^TA^{-1}(AX+BY)=0 \Rightarrow -B^TX - B^TA^{-1}BY = 0$$

$$B^TX + CY = -B^TA^{-1}$$

$$\Rightarrow \underbrace{(C-B^TA^{-1}B)}_{E}Y = \underbrace{-B^TA^{-1}}_{F^T}.$$

Just a sketch.

Remark. This result canbe generalized to arbitrary matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Isometry. A matrix M is called an isometry if $\|Mx\|_2^2 = \|x\|_2^2$.

- M is full rank.
- $\lambda(M)$ an eigenvalue of $M \Rightarrow Mv = \lambda v$

$$\Rightarrow |\lambda(M)| = 1.$$

- $\langle Mx, Mx\rangle = \langle x, x\rangle$
- $|\det(M)| = 1$, what about $\det(M)$? (Exercise)