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Tutorial 1

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Problem 1. *(Basic Entropy)*

Consider two random experiments with three outcomes respectively. The corresponding probability mass functions (e.m.f) are given by

a) (0.9,0.05,0.05)

b) (0.4,0.3,0.3)

Find the entropy for a) and b)?

Problem 2. *(Minimum entropy)*

Let $X \in \{1, \dots, n\}$ be a random variable and $p_i = P(X = i)$. What is the minimum value of $H(X) = H(p_1, ..., p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s that achieve this minimum.

Problem 3. *(Joint entropy)*

Let the joint distribution of two random variables X and Y, denoted by $p(x, y)$, be given in the following table:

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find

a) $H(X)$, $H(Y)$,

b) $H(X|Y)$, $H(Y|X)$,

c) $H(X, Y)$,

d) $H(Y) - H(Y|X)$.

Problem 4. *(Entropy of functions of a random variable)*

Let X be a discrete random variable. Show that the entropy of a deterministic function of X is less than or equal to the entropy of X by justifying the following steps:

$$\begin{aligned} H(X, g(X)) &\stackrel{(a)}{=} H(X) + H(g(X)|X) \\ &\stackrel{(b)}{=} H(X), \\ H(X, g(X)) &\stackrel{(c)}{=} H(g(X)) + H(X|g(X)) \\ &\stackrel{(d)}{\geq} H(g(X)). \end{aligned}$$

Thus, $H(X) \geq H(g(X))$.

Problem 5. *(Entropy of functions)*

Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if

a) $Y = 2^X$?

b) $Y = cosX$?

Problem 6. *(Coin flips)*

A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy $H(X)$in bits.
Hint: For $0 < r < 1$ we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$