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Tutorial 5

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Problem 1. *2 state homogeneous Markov chain*

Consider a 2 state homogeneous Markov chain with states $\{0, 1\}$ and transition matrix $\Pi = \begin{pmatrix} 1-\alpha & 1-\beta \\ \beta & \alpha \end{pmatrix}$.

Compute a stationary distribution $\mathbf{p} = (p_1, p_2)$.

Hint: solve $\mathbf{p}\Pi = \mathbf{p}$

Problem 2. *Recurrence times are insensitive to distributions*

Let $X_0, X_1, X_2, ...$ are drawn i.i.d $\sim p(x), x \in \mathcal{X} = \{1, 2, 3..., m\}$, and let N be the waiting time to the next occurrence of X_0. Thus $N = \min_n\{X_n = X_0\}$.

a) Show that $EN = m$

b) Show that $E \log N \leq H(X)$.

Hint: For $0 < r < 1$ we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$

Problem 3. *Entropy rate of a dog looking for a bone.*

A dog walks on the integers, possibly reversing direction at each step with probability $p = 0.1$. Let $X_0 = 0$. The first step is equally like to be positive or negative. A typical walk might look like this :

$$(X_0, X_1, ...) = (0, -1, -2, -3, -4, -3, -2, -1, 0, 1, ...) \tag{1}$$

a) Find $H(X_1, X_2, ..., X_n)$.

b) Find the entropy rate of the dog.

c) What is the expected number of steps that the dog takes before reversing direction?

Problem 4. *AEP*

Let X_i be i.i.d $\sim p(x), x \in \mathcal{X} = \{1, 2, 3..., m\}$. Let $\mu = EX$ and $H = -\sum p(x) \log p(x)$. Let $A_\epsilon^n = \{(x_1, x_2, ..., x_n) \in \mathcal{X}^n : |-\frac{1}{n} \log p(x^n) - H| \le \epsilon\}$ and $B_\epsilon^n = \{(x_1, x_2, ..., x_n) \in \mathcal{X}^n : |\frac{1}{n}\sum_{i=1}^n x_i - \mu| \le \epsilon\}$.

a) Does $P((X_1, X_2, ..., X_n) \in A_\epsilon^n) \to 1$?

b) Does $P((X_1, X_2, ..., X_n) \in A_\epsilon^n \cap B_\epsilon^n) \to 1$?

c) Show that $|A_\epsilon^n \cap B_\epsilon^n| \le 2^{n(H+\epsilon)}$ for all n.

d) Show that $|A_\epsilon^n \cap B_\epsilon^n| \ge (\frac{1}{2})2^{n(H-\epsilon)}$ for sufficiently large n.

Problem 5. *AEP and mutual information*

Let (X_i, Y_i) be i.i.d. $\sim p(x, y)$. We form the log likelihood ratio of the hypothesis that X and Y are independent vs. the hypothesis that X and Y are dependent, i.e.,

$$\frac{1}{n} \log \frac{p(X_1, X_2, ..., X_n)p(Y_1, Y_2, ..., Y_n)}{p(X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n)}.$$

What is the limit of this log likelihood ratio, when $n \to \infty$?