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Tutorial 7

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Problem 1. *AEP and source coding*

A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary codeword is provided for every sequence of 100 digits containing three or fewer ones.

a) Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequences with three or fewer ones.

b) Calculate the probability of observing a source sequence for which no codeword has been assigned.

c) Use Chebyshev's inequality to bound the probability of observing a source sequence for which no codeword has been assigned. Compare this bound with the actual probability computed in part **b)**.

Problem 2. *Shannon codes and Huffman codes*

Consider a random variable X which takes on four values with probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12})$.

a) Construct a Huffman code for this random variable.

b) Show that there exist two different sets of optimal lengths for the codewords, namely, show that codeword length assignments $(1, 2, 3, 3)$ and $(2, 2, 2, 2)$ are both optimal.

c) Conclude that there are optimal codes with codeword lengths for some symbols that exceed the Shannon code length $\lceil \log \frac{1}{p(x)} \rceil$.

Problem 3. *Twenty Questions*

Player A chooses some object in the universe, and player B attempts to identify the object with a series of yes-no questions. Suppose that player B is clever enough to use the code achieving the minimal expected length with respect to player A's distribution. We observe that player B requires an average of 38.5 questions to determine the object. Find a rough lower bound to the number of objects in the universe.

Problem 4. *Bad Wine*

One is given 6 bottles of wine. It is known that precisely one bottle has gone bad (tastes terrible). From inspection of the bottles it is determined that the probability p_i that the i-th bottle is bad is given by $(p_1, p_2, ..., p_6) = (\frac{8}{23}, \frac{6}{23}, \frac{4}{23}, \frac{2}{23}, \frac{2}{23}, \frac{1}{23})$. Tasting will determine the bad wine. Suppose you taste the wines one at a time. Choose the order of tasting to minimize the expected number of tastings required to determine the bad bottle. Remember, if the first 5 wines pass the test you don't have to taste the last.

a) What is the expected number of tastings required?

b) Which bottle should be tasted first?

Now you get smart. For the first sample, you mix some of the wines in a fresh glass and sample the mixture. You proceed, mixing and tasting, stopping when the bad bottle has been determined.

c) What is the minimum expected number of tastings required to determine the bad wine?

d) What mixture should be tasted first?

Problem 5. *Horse Race*

Three horses run a race. A gambler offers 3-for-1 odds on each of the horses. These are fair odds under the assumption that all horses are equally likely to win the race. The true win probabilities are known to be

$$\mathbf{p} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}).$$

Let $\mathbf{b} = (b_1, b_2, b_3)$, $b_i \geq 0$, $\sum b_i = 1$, be the amount invested on each of the horses. The expected log wealth is thus

$$W(\mathbf{b}) = \sum_{i=1}^{3} p_i \log 3b_i \, .$$

a) Maximize this over $\mathbf{b}$ to find $\mathbf{b}^*$ and W^*. Thus the wealth achieved in repeated horse races should grow to infinity like 2^{nW^*} with probability one.

b) Show that if instead we put all of our money on horse 1, the most likely winner, we will eventually go broke with probability one.