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Tutorial 7

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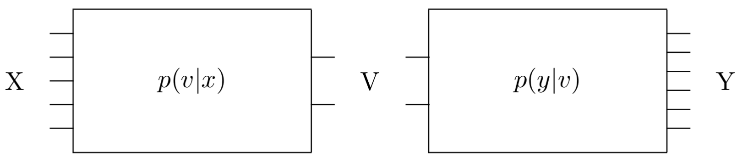
Problem 1. *One bit quantization of a single Gaussian random variable*

Let $X \sim \mathcal{N}(0, \sigma^2)$ and let the distortion measure be squared error. Here we do not allow block descriptions. Show that the optimum reproduction points for 1 bit quantization are $\pm\sqrt{\frac{2}{\pi}}\sigma$, and that the expected distortion for 1 bit quantization is $\frac{\pi-2}{\pi}\sigma^2$. Compare this distortion with the rate distortion bound $D = \sigma^2 2^{-2R}$ for $R = 1$.

Problem 2. *Bottleneck Channel*

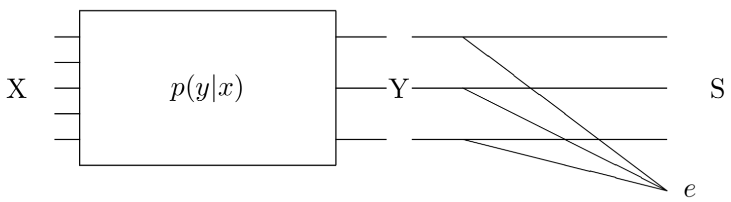
Suppose a signal $X \in \mathcal{X} = 1, 2, \ldots, m$ goes through an intervening transition $X \to V \to Y$:



where $x = \{1, 2, \ldots, m\}, y = \{1, 2, \ldots, m\}$, and $v = \{1, 2, \ldots, k\}$. Here $p(v|x)$ and $p(y|v)$ are arbitrary and the channel has transition probability $p(y|x) = \sum_v p(v|x)p(y|v)$. Show that $C \leq \log k$

Problem 3. *Erasure Channel*

Let $\{\mathcal{X}, p(y|x), \mathcal{Y}\}$ be a discrete memoryless channel with capacity C. Suppose this channel is immediately cascaded with an erasure channel $\{\mathcal{Y}, p(s|y), \mathcal{S}\}$ that erases an α number of symbols.



Specificaslly, $\mathcal{S} = \{y_1, \ldots, y_m, e\}$ and

$$P(S = y|X = x) = (1 - \alpha)p(y|x)\,, \quad y \in \mathcal{Y}$$
$$P(S = e|X = x) = \alpha\,.$$

Determine the capacity of this channel.

Problem 4. *Multiplier Channel*

a) Consider the channel $Y = XZ$ where X and Z are independent binary random variables that take on values 0 and 1. Z is a Bernoulli(α), i.e. $P(Z = 1) = \alpha$. Find the capacity of this channel and the maximizing distribution on X .

b) Now suppose the receiver can observe Z as well as Y. What is the capacity?

Problem 5. *A channel with two independent looks at Y*

Let Y_1 and Y_2 be conditionally indeopendent and conditionally identically distributed given X.

a) Show that $I(X;Y_1,Y_2) = 2I(X;Y_1) - I(Y_1;Y_2)$.

b) Conclude that the capacity of the channel

$$X \to \boxed{\text{Channel}} \to (Y_1, Y_2)$$

is less than twice the capacity of the channel

$$X \to \boxed{\text{Channel}} \to Y_1 \,.$$

Problem 6. *Bounds on the rate distortion function for squared error distortion.*

For the case of a continuous random variable X with mean zero and variance σ^2 and squared error distortion, show that

a) $h(X) - \frac{1}{2}\log(2\pi e D) \leq R(D)$,

b) $R(D) \leq \frac{1}{2}\log\frac{\sigma^2}{D}$.