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Tutorial 1
- Proposed Solution -

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Solution of Problem 1

(Conditional Entropy)

Let X, Y, Z be discrete random variables. Proof that:

a) $0 \overset{(i)}{\leq} H(X|Y) \overset{(ii)}{\leq} H(X)$.

- Equality holds in $(i) \Leftrightarrow X$ is totally dependent on Y.
- Equality holds in $(ii) \Leftrightarrow X$ and Y are independent.

Equality (i) holds iff $H(X|Y) = 0$, that is if X is a deterministic function of Y. Similarly, (ii) holds iff $H(X) = H(X|Y)$. Since

$$H(X|Y) = H(X) - I(X;Y)$$

we have that this holds only if $I(X;Y) = 0$, that is X and Y are statistically independent.

b) $H(X|Y,Z) \leq \min\{H(X|Y), H(X|Z)\}$.

$$H(X|Y,Z) = H(X|Z) - \underbrace{I(X;Y|Z)}_{\geq 0} \leq H(X|Z)\,.$$

The same can be said for $H(X|Y,Z) \leq H(X|Y)$.

Solution of Problem 2

(Sequence of Random Variables)

Let $X_0 \in \mathcal{X}$ be a discrete random variable with distribution μ_0. Given the stochastic matrix Π, let $X_1, X_2, \ldots$ be the sequence of random variables (with the same support $\mathcal{X}$) such that μ_n is the distribution[1] of X_n and $\mu_n = \mu_{n-1}\Pi$ for all $n = 1, 2, \ldots$.

a) Show that $\mu_n = \mu_0\Pi^n$.

$$\mu_n = \mu_{n-1}\Pi = (\mu_{n-2}\Pi)\Pi = \cdots = \mu_0 \underbrace{\Pi\Pi\cdots\Pi}_{n \text{ times}} = \mu_0\Pi^n\,.$$

[1] Note that μ_n are row vectors for all $n = 0, 1, 2, \ldots$.

Now assume that μ_n converges to some distribution μ^*, that is $\lim_{n\to\infty} \mu_n = \mu^*$ and $\mu^*\Pi = \mu^*$.

b) Proof that $D(\mu_n\|\mu^*) \geq D(\mu_{n+1}\|\mu^*)$ for all $n = 1, 2, \ldots$.

$$D(\mu_n\|\mu^*) = D(\mu_{n-1}\Pi\|\mu^*\Pi) \overset{(2.1.13)}{\leq} D(\mu_{n-1}\|\mu^*)\,.$$

c) Show that if μ^* is the uniform distribution then $H(X_n) \leq H(X_{n+1})$ for all $n = 1, 2, \ldots$.

Since μ_* is the uniform distribution we have

$$D(\mu_n\|\mu^*) = \log|\mathcal{X}| - H(X_n)\,. \quad \text{(see proof of theorem 2.1.14)}$$

Then we get

$$\begin{aligned} D(\mu_n\|\mu^*) &\geq D(\mu_{n+1}\|\mu^*) \\ \Rightarrow \log|\mathcal{X}| - H(X_n) &\geq \log|\mathcal{X}| - H(X_{n+1}) \\ \Rightarrow H(X_n) &\leq H(X_{n+1})\,. \end{aligned}$$

Solution of Problem 3

(A Metric)

A function $\rho(x, y)$ is a metric if for all x, y,

- $\rho(x, y) \geq 0$,
- $\rho(x, y) = \rho(y, x)$,
- $\rho(x, y) = 0$ if and only if $x = y$,
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

a) Show that $\rho(X, Y) = H(X|Y) + H(Y|X)$ satisfies the first, second and fourth properties above. If we say that $X = Y$ if there is a one-to-one function mapping from X to Y, then the third property is also satisfied, and $\rho(X, Y)$ is a metric.

- Since conditional entropies are non-negative we have that $\rho(X, Y) \geq 0$.
- By definition $\rho(X, Y) = \rho(Y, X)$.
- $H(X|Y) = 0$ iff there is a deterministic mapping from Y to X. In the same manner $H(Y|X) = 0$ iff there is a deterministic mapping from X to Y. Therefore, $H(X|Y) = 0$ and $H(X|Y) = 0$ only if there is a one-to-one mapping between X and Y.
- $$\begin{aligned} H(X|Y) + H(Y|Z) &\geq H(X|Y, Z) + H(Y|Z) \\ &= H(X|Y, Z) \\ &= H(X|Z) + H(Y|X, Z) \\ &\geq H(X|Z)\,. \end{aligned}$$

  Then it follows $\rho(X, Y) + \rho(Y, Z) \geq \rho(X, Z)$.

b) Verify that $\rho(X,Y)$ can also be expressed as

$$\begin{aligned}\rho(X,Y) &= H(X)+H(Y)-2I(X;Y)\\ &= H(X,Y)-I(X;Y)\\ &= 2H(X,Y)-H(X)-H(Y)\end{aligned}$$

$$\begin{aligned}\rho(X,Y) &= \underbrace{H(X|Y)}_{=H(X)-I(X;Y)} + \underbrace{H(Y|X)}_{=H(Y)-I(X;Y)} = H(X)+H(Y)-2I(X;Y)\\ &= \underbrace{H(X)+H(Y)-I(X;Y)}_{H(X,Y)} - I(X;Y) = H(X,Y)-I(X;Y)\\ &= H(X,Y)\underbrace{-I(X;Y)+H(X)+H(Y)}_{H(X,Y)}-H(X)-H(Y) = 2H(X,Y)-H(X)-H(Y)\,.\end{aligned}$$

Solution of Problem 4

(A Measure of Correlation)

Let X_1 and X_2 be identically distributed random variables, but not necessarily independent. Let

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}\,.$$

a) Show that $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.

Since X_1 and X_2 be identically distributed, we have that $H(X_1) = H(X_2)$. Then

$$\rho = \frac{H(X_1)-H(X_2|X_1)}{H(X_1)} = \frac{H(X_2)-H(X_2|X_1)}{H(X_1)} = \frac{I(X_1;X_2)}{H(X_1)}\,.$$

b) Show that $0 \le \rho \le 1$.

Since $I(X_1;X_2)$ and $H(X_1)$ are non-negative $\rho \ge 0$. On the other hand

$$I(X_1;X_2) = H(X_1) - \underbrace{H(X_1|X_2)}_{\ge 0} \le H(X_1) \;\Rightarrow\; \rho \le 1\,.$$

c) When is $\rho = 0$?

$\rho = 0 \Leftrightarrow I(X;Y) = 0 \Leftrightarrow X, Y$ are independent.

d) When is $\rho = 1$?

$\rho = 1 \Leftrightarrow H(X_1|X_2) = 0$, by symmetry $\rho = 1 \Leftrightarrow H(X_2|X_1) = 0$. Therefore, $\rho = 1 \Leftrightarrow X, Y$ have a one-to-one relationship.

Solution of Problem 5

(Entropy of a Sum)

Let X and Y be random variables that take on values $x_1, x_2, \dots, x_r$ and $y_1, y_2, \dots, y_r$ respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.

Since $Z = X + Y$ we have

$$P(Z = z|X = x) = P(Y = z - x|X = x)\,.$$

Then,

$$\begin{aligned}
H(Z|X) &= \sum_{x\in\mathcal{X}} P(X = x)H(Z|X = x) \\
&= -\sum_{x\in\mathcal{X}} P(X = x)\sum_{z\in\mathcal{Z}} P(Z = z|X = x)\log P(Z = z|X = x) \\
&= -\sum_{x\in\mathcal{X}} P(X = x)\sum_{z\in\mathcal{Z}} P(Y = \underbrace{z - x}_{y}|X = x)\log P(Y = z - x|X = x) \\
&= -\sum_{x\in\mathcal{X}} P(X = x)\sum_{y\in\mathcal{Y}} P(Y = y|X = x)\log P(Y = y|X = x) \\
&= \sum_{x\in\mathcal{X}} P(X = x)H(Y|X = x) \\
&= H(Y|X)\,.
\end{aligned}$$

b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

Let X be uniformly distributed and $Y = -X$, thus Y is also uniform. Then $Z = X + Y = 0$, thus $H(Z) = 0$ while $H(X) = \log|\mathcal{X}| > 0$ and $H(Y) = \log|\mathcal{Y}| > 0$.

c) Under what conditions does $H(Z) = H(X) + H(Y)$?

$$H(Z) \underset{(i)}{\leq} H(X,Y) \underset{(ii)}{\leq} H(X) + H(Y)\,.$$

$H(Z) = H(X) + H(Y)$ iff $(i), (ii)$ hold with equality, that is

- (i): (X,Y) has a one-to-one relation with Z,
- (ii): X and Y are independent.