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Tutorial 4
- Proposed Solution -

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Solution of Problem 1

(Chain Rule)

Let $X_1, \dots, X_n$ and Y be discrete random variables. Proof that

a) $H(X_1, \dots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \dots, X_1)$.

By the two-variable expansion rule of entropies we have

$$\begin{aligned} H(X_1, X_2) &= H(X_1) + H(X_2|X_1) \\ H(X_1, X_2, X_3) &= H(X_1) + H(X_2, X_3|X_1) = H(X_1) + H(X_2|X_1) + H(X_3|X_2, X_1) \\ &\vdots \\ H(X_1, X_2, \dots, X_n) &= \sum_{i=1}^{n} H(X_i|X_{i-1}, \dots, X_1) \,. \qquad \square \end{aligned}$$

b) $I(X_1, \dots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y|X_{i-1}, \dots, X_1)$.

$$\begin{aligned} I(X_1, \dots, X_n; Y) &= H(X_1, \dots, X_n) - H(X_1, \dots, X_n|Y) \\ &= \sum_{i=1}^{n} H(X_i|X_{i-1}, \dots, X_1) - \sum_{i=1}^{n} H(X_i|X_{i-1}, \dots, X_1, Y) \\ &= \sum_{i=1}^{n} [H(X_i|X_{i-1}, \dots, X_1) - H(X_i|X_{i-1}, \dots, X_1, Y)] \\ &= \sum_{i=1}^{n} I(X_i; Y|X_{i-1}, \dots, X_1) \,. \qquad \square \end{aligned}$$

Let $Z = \{(X_n, Y_n)\}_{n \in \mathbb{N}}$ be an i.i.d. sequence of pairs of discrete random variables.

c) Show that $I(X_1, \dots, X_n; Y_1, \dots, Y_n) = \sum_{i=1}^{n} I(X_i; Y_i)$.

$$\begin{aligned} I(X_1, \dots, X_n; Y_1, \dots, Y_n) &= \sum_{i=1}^{n} I(X_i; Y_1, \dots, Y_n|X_{i-1}, \dots, X_1) \\ &= \sum_{i=1}^{n} I(X_i; Y_i|X_{i-1}, \dots, X_1) \\ &= \sum_{i=1}^{n} I(X_i; Y_i) \,. \qquad \square \end{aligned}$$

Solution of Problem 2

(Stationary Processes)

Let $\dots, X_{-1}, X_0, X_1, \dots$ be a stationary stochastic process. Which of the following are true? Prove or provide a counterexample.

a) $H(X_n|X_0) = H(X_{-n}|X_0)$.

This is true since

$$\begin{aligned} H(X_n|X_0) &= H(X_n, X_0) - H(X_0) \\ H(X_{-n}|X_0) &= H(X_{-n}, X_0) - H(X_0) \end{aligned}$$

and $H(X_n, X_0) = H(X_{-n}, X_0)$ by stationarity.

b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$.

This is not true in general. A simple couterexample would be a periodic process with period n. Let $X_0, \dots, X_{n-1}$ be i.i.d. uniform random variables and let $X_k = X_{k-n}$ for $k \geq n$. In this case $H(X_n|X_0) = 0$ and $H(X_{n-1}|X_0) = \log|\mathcal{X}|$, contradicting the statement.

c) $H(X_n|X_1, X_2, \dots, X_{n-1}, X_{n+1})$ is non-increasing in n.

This statement is true, since by stationarity

$$\begin{aligned} H(X_n|X_1, X_2, \dots, X_{n-1}, X_{n+1}) &= H(X_{n+1}|X_2, X_3, \dots, X_n, X_{n+2}) \\ &\geq H(X_{n+1}|X_1, X_2, X_3, \dots, X_n, X_{n+2}) \,. \end{aligned}$$

d) $H(X_n|X_1, \dots, , X_{n-1}, X_{n+1}, \dots, X_{2n})$ is non-increasing in n.

In the same manner, this statement is true since

$$\begin{aligned} H(X_n|X_1, X_2, \dots, X_{n-1}, X_{n+1}, \dots, X_{2n}) &= H(X_{n+1}|X_2, X_3, \dots, X_n, X_{n+2}, \dots, X_{2n+1}) \\ &\geq H(X_{n+1}|X_1, X_2, X_3, \dots, X_n, X_{n+2}, \dots, X_{2n+1}) \,. \end{aligned}$$

Solution of Problem 3

(The past has little to say about the future)

For a stationary stochastic process $X_1, X_2, \dots,$ show that

$$\lim_{n\to\infty} \frac{1}{2n} I(X_1, \dots, X_n; X_{n+1}, \dots, X_{2n}) = 0 \,.$$

Since $X_1, X_2, \dots$ is stationary we know that $H(X_{n+1}, \dots X_{2n}) = H(X_1, \dots, X_n)$, then

$$\begin{aligned} &I(X_1, \dots, X_n; X_{n+1}, \dots X_{2n}) \\ &\quad = H(X_1, \dots, X_n) + \underbrace{H(X_{n+1}, \dots X_{2n})}_{=H(X_1,\dots,X_n)} - H(X_1, \dots, X_n, X_{n+1}, \dots X_{2n}) \\ &\quad = 2H(X_1, \dots, X_n) - H(X_1, \dots, X_n, X_{n+1}, \dots X_{2n}) \,. \end{aligned}$$

Thus

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2n} I(X_1,\dots,X_n;X_{n+1},\dots X_{2n}) \\
&\qquad = \lim_{n\to\infty} \frac{1}{2n} 2H(X_1,\dots,X_n) - \lim_{n\to\infty} \frac{1}{2n} H(X_1,\dots,X_n,X_{n+1},\dots X_{2n}) \\
&\qquad = \underbrace{\lim_{n\to\infty} \frac{1}{n} H(X_1,\dots,X_n)}_{=H_\infty(X)} - \underbrace{\lim_{n\to\infty} \frac{1}{2n} H(X_1,\dots,X_n,X_{n+1},\dots X_{2n})}_{=H_\infty(X)} \\
&\qquad = 0\,. \qquad \square
\end{aligned}
$$

Solution of Problem 4

(Entropy Rate)

Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stationary sequence of discrete random variables with entropy rate $H_\infty(X)$.

a) Show that $H_\infty(X) \le H(X_1)$.

From Theorem 2.3.4 (d) we get

$$
\begin{aligned}
H_\infty(X) &= \lim_{n\to\infty} H(X_n|X_0,\dots,X_{n-1}) \\
&= \lim_{n\to\infty} H(X_1|X_{-(n-1)},\dots,X_{-1},X_0) \\
&\le \lim_{n\to\infty} H(X_1) = H(X_1)\,. \qquad \square
\end{aligned}
$$

b) What are the conditions for equality?

We have equality only if X_1 is independent of the past $X_0, X_{-1}, \dots$, i.e., if and only if X_i is an i.i.d. process.