RWTH AACHEN UNIVERSITY

Theorem (p.39)

B discrete r.v., independent with support $\{0,\dots,K-1\}$

$\tilde{Y} = c(X)$ true class label

$Y = \tilde{Y} \oplus B = \tilde{Y} + B \mod K$

Then

$$\varepsilon = R_{\alpha}(g) = P(\hat{Y} \neq Y) \geq \phi(H(B))$$

Proof. $H(Y|\hat{Y}) \geq H(Y|\hat{Y}, \tilde{Y}) = H(B|\hat{Y}, \tilde{Y}) = H(B)$

By Fano's inequality

$$H(Y|\hat{Y}) \leq \psi(R(g_\alpha))$$

Hence

$$\psi(R(g_\alpha)) \geq H(B)$$

and

$$R(g_\alpha) \geq \phi(H(B))$$

Special case

$$P(B=i) = \begin{cases} 1-p & , i=0 \\ \frac{p}{K-1} & , i=1,\dots,K-1 \end{cases} , \quad p \le 1-\frac{1}{K}$$

Then

$$H(B) = -p \log p - (1-p)\log(1-p) + p \log(K-1)$$
$$= \Psi(p)$$

Hence

$$R(g_{\omega}) \geq \Phi(\Psi(p)) = p$$

Theorem p.41 If $R(g_\omega) = p$ then
$$P(\hat{Y} = \tilde{Y}) = 1 \quad \lrcorner$$

Proof. Set $P(\hat{Y}=\tilde{Y}) = \delta$

$$\begin{aligned} 1-p &= P(\hat{Y} = Y) \\ &= \sum_{i=0}^{K-1} \{ P(\hat{Y}=Y, \hat{Y}=\tilde{Y}, B=i) + P(\hat{Y}=Y, \hat{Y}\neq\tilde{Y}, B=i \} \\ &= P(\hat{Y}=\tilde{Y}, B=0) + \sum_{i=1}^{K-1} P(\hat{Y}=\tilde{Y}\oplus i, \hat{Y}\neq\tilde{Y}, B=i) \\ &= P(B=0)\delta + \sum_{i=1}^{K-1} P(\hat{Y}=\tilde{Y}\oplus i, B=i) \\ &= (1-p)\delta + \frac{p}{K-1} \underbrace{\sum_{i=1}^{K-1} P(\hat{Y}=\tilde{Y}\oplus i)}_{=1-\delta} \\ &= (1-p)\delta + \frac{p}{K-1}(1-\delta) \end{aligned}$$

Hence $(1-p)(1-\delta) = \frac{p}{K-1}(1-\delta)$

It folls $\delta = 1$. $\blacksquare$