Homework 4 in Cryptography

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Exercise 13. Let X, Y be discrete random variables on a set Ω.

a) Show that for any function $f : X(\Omega) \times Y(\Omega) \to \mathbb{R}$, the relationship
$$H(X, Y, f(X, Y)) = H(X, Y)$$
holds.

Exercise 14. Let $(\mathcal{M}, \mathcal{K}, \mathcal{C}, e, d)$ be a cryptosystem. Suppose that $P(\hat{M} = M) > 0$ for all $M \in \mathcal{M}$, $P(\hat{K} = K) > 0$ for all $K \in \mathcal{K}$ and $|\mathcal{M}| = |\mathcal{K}| = |\mathcal{C}|$ holds. Show that if $(\mathcal{M}, \mathcal{K}, \mathcal{C}, e, d)$ has perfect secrecy, then
$$P(\hat{K} = K) = \frac{1}{|\mathcal{K}|} \text{ for all } K \in \mathcal{K}$$
and for all $M \in \mathcal{M}, C \in \mathcal{C}$, there is a unique $K \in \mathcal{K}$ such that $e(M, K) = C$.

Exercise 15. Let $\mathcal{M} = \{a, b\}$ be the message space, $\mathcal{K} = \{K_1, K_2, K_3\}$ the key space and $\mathcal{C} = \{1, 2, 3, 4\}$ the ciphertext space. Let $\hat{M}, \hat{K}$ be stochastically independent random variables with support $\mathcal{M}$ and $\mathcal{K}$, respectively, and with probability distributions:
$$P(\hat{M} = a) = \frac{1}{4},\; P(\hat{M} = b) = \frac{3}{4},\; P(\hat{K} = K_1) = \frac{1}{2},\; P(\hat{K} = K_2) = \frac{1}{4},\; P(\hat{K} = K_3) = \frac{1}{4}.$$

The following table explains the encryption rules:

	K_1	K_2	K_3
a	1	2	3
b	2	3	4

, e.g., $e(a, K_1) = 1$.

a) Compute the entropies $H(\hat{M}), H(\hat{K}), H(\hat{C})$ and the key equivocation $H(\hat{K} \mid \hat{C})$.

b) Why does this cryptosystem not have perfect secrecy?

Exercise 16. Consider affine ciphers on $\mathbb{Z}_{26}$, i.e., $\mathcal{M} = \mathcal{C} = \mathbb{Z}_{26}$ and $\mathcal{K} = \mathbb{Z}_{26}^* \times \mathbb{Z}_{26} = \{(a, b) \mid a, b \in \mathbb{Z}_{26}, \gcd(a, 26) = 1\}$. Select the key $\hat{K}$ uniformly distributed at random and independently from the message $\hat{M}$.

Show that this cryptosystem has perfect secrecy.