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Exercise 13
- Proposed Solution -

Friday, July 28, 2017

Solution of Problem 1

a) $p = 13$ is a prime number, $a = 5$ is a quadratic residue mod p.

1) $v = b^2 - 4a = b^2 - 4 \cdot 5 = b^2 - 20.$

   Choose: $b = 5 \Rightarrow v = 25 - 20 = 5.$
   With Euler's criterion, compute: $(\frac{v}{p}) = (\frac{5}{11}) = 5^{\frac{10}{2}} = 1.$
   $\Rightarrow v = 5$ is a quadratic residue mod 11. ↯

   Choose: $b = 6 \Rightarrow v = 36 - 20 = 16 \equiv 5 \mod 11.$
   $\Rightarrow v = 5$ is a quadratic residue mod 11. ↯

   Choose: $b = 7 \Rightarrow v = 49 - 20 = 29 \equiv 7 \mod 11.$
   With Euler's criterion, compute:
   $(\frac{7}{11}) = 7^{\frac{p-1}{2}} \equiv 7^{\frac{10}{5}} \equiv 7^5 \equiv 49 \cdot 49 \cdot 7 \equiv 5 \cdot 5 \cdot 7 \equiv -1 \mod 11.$
   $\Rightarrow v$ is a quadratic non-residue modulo 11. ✓

2) Insert the values for a and b into the polynomial $f(x) = x^2 - 7x + 5$.

3) Compute $r = x^{\frac{p+1}{2}} \mod f(x)$:

$$
\begin{array}{l}
x^6 : (x^2 - 7x + 5) = x^4 + 7x^3 + 2x - 3 \\
-\,(x^6 - 7x^5 + 5x^4) \\
\hline
+\,7x^5 - 5x^4 \\
-\,(7x^5 - 5x^4 + 2x^3) \\
\hline
-\,2x^3 \\
-\,(-2x^3 + 3x^2 - 10x) \\
\hline
-\,3x^2 + 10x \\
-\,(-3x^2 + 10x - 4) \\
\hline
4
\end{array}
$$

Hence, $r = 4$. Furthermore, and $-r = -4 \equiv 7 \mod 11 \Rightarrow (r, -r) = (4, 7)$.
// Validation $r^2 = a \mod 11$ is correct in both cases.

b) Both p, q satisfy the requirement for a Rabin cryptosystem: $p, q \equiv 3 \mod 4$.
For $c \mod p \equiv 225 \mod 11 \equiv 5$, we already know the square roots $x_{p,1} = 4$, $x_{p,2} = 7$.

For $c \mod q \equiv 225 \mod 23 \equiv 18$, compute the square roots $x_{q,1}$, $x_{q,2}$ with the auxiliary parameter $k_q = \frac{q+1}{4} = 6$:

$$\begin{aligned} x_{q,1} &= c^{k_q} = 18^6 = 18^3 \cdot 18^3 \equiv 13 \cdot 13 \equiv 8 \mod 23, \\ x_{q,2} &= -8 \equiv 15 \mod 23. \end{aligned}$$

Formulate $tq + sp = 1$:

$$\begin{aligned} 23 &= 2 \cdot 11 + 1 \\ \Rightarrow 1 &= 23 - 2 \cdot 11 \end{aligned}$$

We set $a = tq = 23$ and $b = sp = -22$. Compute all four possible solutions:

$$\begin{aligned} m_{11} &= ax_{p,1} + bx_{q,1} = 23 \cdot 4 - 22 \cdot 8 = -84 \equiv 169 \mod 253 \Rightarrow (...1001)_2 \quad ↯ \\ m_{12} &= ax_{p,1} + bx_{q,2} = 23 \cdot 4 - 22 \cdot 15 = -238 \equiv 15 \mod 253 \Rightarrow (...1111)_2 \quad ↯ \\ m_{21} &= ax_{p,2} + bx_{q,1} = 23 \cdot 7 - 22 \cdot 8 = -15 \equiv 238 \mod 253 \Rightarrow (...1110)_2 \quad ↯ \\ m_{22} &= ax_{p,2} + bx_{q,2} = 23 \cdot 7 - 22 \cdot 15 = -169 \equiv 84 \mod 253 \Rightarrow (...0100)_2 \quad \checkmark \end{aligned}$$

The solution is $m = m_{21} = 84$ since it ends on 0100 in the binary representation.
// Checking all solutions yields $c = 225$.

c) Since $c = 225$, one is enabled to compute two square roots in the reals, $m = \pm 15$. If naive Nelson chooses 1111, the result $m = 15$ is obvious, without knowing the factors in $n = pq$.

Solution of Problem 2

Decipher $m = \sqrt{c} \mod n$ with $c = 1935$.

- Check $p, q \equiv 3 \mod 4$ ✓
- Compute the square roots of c modulo p and c modulo q.

$$\begin{aligned}
k_p &= \frac{p+1}{4} = 17, \quad k_q = \frac{q+1}{4} = 18, \\
x_{p,1} &= c^{k_p} \equiv 1935^{17} \equiv 59^{17} \equiv 40 \mod 67, \\
x_{p,2} &= -x_{p,1} \equiv 27 \mod 67, \\
x_{q,1} &= c^{k_q} \equiv 1935^{18} \equiv 18^{18} \equiv 36 \mod 71, \\
x_{q,2} &= -x_{q,1} \equiv 35 \mod 71.
\end{aligned}$$

- Compute the resulting square root modulo n. $m_{i,j} = ax_{p,i} + bx_{q,j}$ solves $m_{i,j}^2 \equiv c \mod n$ for $i, j \in \{1, 2\}$. We substitute $a = tq$ and $b = sp$. Then $tq + sp = 1$ yields $1 = 17 \cdot 71 + (-18) \cdot 67 = tq + sp$ from the Extended Euclidean Algorithm.

$$\begin{aligned}
&\Rightarrow a \equiv tq \equiv 17 \cdot 71 \equiv 1207 \mod n \\
&\Rightarrow b \equiv -sp \equiv -18 \cdot 67 \equiv -1206 \mod n.
\end{aligned}$$

The four possible solutions for the square root of ciphertext c modulo n are:

$$\begin{aligned}
m_{1,1} &\equiv ax_{p,1} + bx_{q,1} \equiv 107 \mod n \Rightarrow 00000011010\underline{11}, \\
m_{1,2} &\equiv ax_{p,1} + bx_{q,2} \equiv 1313 \mod n \Rightarrow 0010100100001, \\
m_{2,1} &\equiv ax_{p,2} + bx_{q,1} \equiv 3444 \mod n \Rightarrow 0110101110100, \\
m_{2,2} &\equiv ax_{p,2} + bx_{q,2} \equiv 4650 \mod n \Rightarrow 1001000101010.
\end{aligned}$$

The correct solution is m_1, by the agreement given in the exercise.

Solution of Problem 3

a) Given $x \equiv -x \mod p$, prove that $x \equiv 0 \mod p$.

Proof. The inverse of 2 modulo p exists. Then,

$$\begin{aligned} & -x \equiv x && \mod p \\ \Leftrightarrow\quad & 0 \equiv 2x && \mod p \\ \Leftrightarrow\quad & 0 \equiv x && \mod p\,. \end{aligned}$$

□

b) Looking at the protocol, we can show that Bob always loses to Alice, if she chooses $p = q$.

- **i)** Alice calculates $n = p^2$ and sends n to Bob.
- **ii)** Bob calculates $c \equiv x^2 \mod n$ and sends c to Alice. With high probability $p \nmid x \Leftrightarrow x \not\equiv 0 \mod p$ (therefore, Bob *almost* always loses).
- **iii)** The only two solutions $\pm x$ are calculated by Alice (see below) and sent to Bob. Bob cannot factor n, as

$$\gcd(x - (\pm x), n) = \begin{cases} \gcd(0, n) = n \\ \gcd(2x, n) = \gcd(2x, p^2) = 1 \end{cases}.$$

  Alice always wins.

c) If Bob asks for the secret key as confirmation, the square is revealed and Alice will be accused of cheating. Bob can factor n by calculating $p = \sqrt{n}$ as a real number and win the game.

Note: The two solutions $\pm x$ to $x^2 \equiv c \mod p^2$ can be calculated as follows.

Let p be an odd prime and $x, y \not\equiv 0 \mod p$. If $x^2 \equiv y^2 \mod p^2$, then $x^2 \equiv y^2 \mod p$, so $x \equiv \pm y \mod p$.

Let $x \equiv y \mod p$. Then

$$x = y + \alpha p\,.$$

By squaring we get

$$\begin{aligned} x^2 &= y^2 + 2\alpha p y + (\alpha p)^2 \\ \Rightarrow x^2 &\equiv y^2 + 2\alpha p y \mod p^2\,. \end{aligned}$$

Since $x^2 \equiv y^2 \mod p^2$, we obtain

$$0 = 2\alpha p y \mod p^2\,.$$

Divide by p to get

$$0 = 2\alpha y \mod p\,.$$

Since p is odd and $p \nmid y$, we must have $p \mid \alpha$. Therefore, $x = y + \alpha p \equiv y \mod p^2$. The case $x \equiv -y \mod p$ is similar.

In other words, if $x^2 \equiv y^2 \mod p^2$, not only $x \equiv \pm y \mod p$, but also $x \equiv \pm y \mod p^2$. At this point, we have shown that only two solutions exist.

Now, we show how to find $\pm x$, where $x^2 \equiv c \mod p^2$. As we can find square roots modulo a prime p, we have $x = b$ solves $x^2 \equiv c \mod p$. We want $x^2 \equiv c \mod p^2$. Square $x = b + ap$ to get

$$\begin{gathered} b^2 + 2bap + (ap)^2 \equiv b^2 + 2bap \equiv c \mod p \\ \Rightarrow b^2 \equiv c \mod p\,. \end{gathered}$$

Since $b^2 \equiv c \mod p$ the number $c - b^2$ is a multiple of p, so we can divide by p and get

$$2ab \equiv \frac{c - b^2}{p} \mod p\,.$$

Multiplying by the multiplicative inverse modulo p of 2 and b, we obtain:

$$a \equiv \frac{c - b^2}{p} \cdot 2^{-1} \cdot b^{-1} \mod p\,.$$

Therefore, we have $x = b + ap$.

This procedure can be continued to get solutions modulo higher powers of p. It is the numberic-theoretic version of Newton's method for numerically solving equations, and is usually referred to as Hensel's Lemma.

Example: $p = 7$, $p^2 = 49$, $c = 37$. Then

$$\begin{gathered} b = c^{\frac{p+1}{4}} = 37^{\frac{7+1}{4}} = 37^2 \equiv 4 \mod p\,, \\ b^{-1} \equiv 2 \mod p\,,\ \ 2^{-1} \equiv 4 \mod p\,, \\ a = \frac{c - b^2}{p} \cdot 2^{-1} \cdot b^{-1} = \frac{37 - 4^2}{7} \cdot 4 \cdot 2 \equiv 3 \mod p \\ x = b + ap = 4 + 3 \cdot 7 = 25 \end{gathered}$$

Check: $x^2 = 25^2 \equiv 37 = c \mod p^2$.