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# Exercise 7
# - Proposed Solution -

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## Solution of Problem 1

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} x & (x+1) & 1 & 1 \\ 1 & x & (x+1) & 1 \\ 1 & 1 & x & (x+1) \\ (x+1) & 1 & 1 & x \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{F}_{2^8}^4 \tag{1}$$

It is to show that:

$$(c_3u^3 + c_2u^2 + c_1u + c_0)((x+1)u^3 + u^2 + u + x) \equiv \sum_{i=0}^{3} r_i u^i \pmod{(u^4+1)}. \tag{2}$$

We expand the multiplication on the left hand side of (2), reduce it modulo $u^4 + 1 \in \mathbb{F}_{2^8}[u]$, and use the abbreviations $(r_0, r_1, r_2, r_3)'$ according to (1).

$$\begin{aligned}
&(c_3u^3 + c_2u^2 + c_1u + c_0)((x+1)u^3 + u^2 + u + x) \\
= &\, c_3(x+1)u^6 + c_3u^5 + c_3u^4 + c_3xu^3 + \\
&\, c_2(x+1)u^5 + c_2u^4 + c_2u^3 + c_2xu^2 + \\
&\, c_1(x+1)u^4 + c_1u^3 + c_1u^2 + c_1xu + \\
&\, c_0(x+1)u^3 + c_0u^2 + c_0u + c_0x \\
= &\, [c_3(x+1)]u^6 + [c_3 + c_2(x+1)]u^5 + [c_3 + c_2 + c_1(x+1)]u^4 \\
&\, + [c_3x + c_2 + c_1 + c_0(x+1)]u^3 + [c_2x + c_1 + c_0]u^2 + [c_1x + c_0]u + c_0x.
\end{aligned}$$

Now, we apply the modulo operation and merge terms:

$$\begin{aligned}
&\equiv [c_3x + c_2 + c_1 + (x+1)c_0]u^3 + [c_3(x+1) + c_2x + c_1 + c_0]u^2 + \\
&\quad\ [c_3 + c_2(x+1) + c_1x + c_0]u + [c_3 + c_2 + c_1(x+1) + c_0x] \\
&\overset{(1)}{\equiv} r_3u^3 + r_2u^2 + r_1u + r_0 \equiv \sum_{i=0}^{3} r_i u^i \pmod{(u^4+1)}
\end{aligned}$$

## Solution of Problem 2

Given: Alphabet $\mathcal{A}$, blocklength $n \in \mathbb{N}$ and $\mathcal{M} = \mathcal{A}^n = \mathcal{C}$.
$\mathcal{A}^n$ describes all possible streams of $n$ bits.

**a)** An encryption is an injective function $e_K : \mathcal{M} \to \mathcal{C}$, with $K \in \mathcal{K}$.
Fix key $K \in \mathcal{K}$. As $e(\cdot, K)$ is injective, it holds:

- $\{e(M, K) \mid M \in \mathcal{M}\} \subseteq \mathcal{C}$
- $\{e(M, K) \mid M \in \mathcal{M}\} = \mathcal{M}$
- Since $\mathcal{M} = \mathcal{C} \Rightarrow e(\mathcal{M}, K) = \mathcal{C}$ also surjective
- $\Rightarrow e(\mathcal{M}, K)$ is a bijective function.

A permutation $\pi$ is a bijective (one-to-one) function $\pi : \mathcal{X} \to \mathcal{X}$.
$\Rightarrow$ For each $K$, the encryption $e(\cdot, K)$ is a permutation with $\mathcal{X} = \mathcal{A}^n$.

**b)** With $\mathcal{A} = \{0, 1\} \Rightarrow |\mathcal{A}| = |\{0, 1\}| = 2$, and $n = 6$ there are $N = 2^6 = 64$ elements. It follows that there are $64! \approx 1.2689 \cdot 10^{89}$ different block ciphers.

## Solution of Problem 3

Let $\varphi : \mathbb{N} \to \mathbb{N}$ the Euler $\varphi$-function, i.e., $\varphi(n) = |\mathbb{Z}_n^*|$ with $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$.

**a)** Let $n = p$ be prime. It follows for the multiplicative group that:

$$\mathbb{Z}_p^* = \{a \in \mathbb{Z}_p \mid \gcd(a, p) = 1\} = \{1, 2, \dots, p-1\} \Rightarrow \varphi(p) = p - 1 \,.$$

**b)** The power $p^k$ has only one prime factor. So $p^k$ has a common divisors that are not equal to one: These are only the multiples of $p$. For $1 \leq a \leq p^k$:

$$1 \cdot p\,, \quad 2 \cdot p\,, \quad \dots\,, \quad p^{k-1} \cdot p = p^k \,.$$

And it follows that

$$\varphi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1) \,.$$

**c)** Let $n = p\,q$ for two primes $p \neq q$. It holds for $1 \leq a < p\,q$

1) $p \mid a \vee q \mid a \Rightarrow \gcd(a, p\,q) > 1$, and
2) $p \nmid a \wedge q \nmid a \Rightarrow \gcd(a, p\,q) = 1$.

It follows $\mathbb{Z}_{p\,q}^* = \underbrace{\{1 \leq a \leq p\,q - 1\}}_{p\,q-1 \text{ elements}} \setminus \left[ \underbrace{\{1 \leq a \leq p\,q - 1 \mid p \mid a\}}_{q-1 \text{ elements}} \,\dot{\cup}\, \underbrace{\{1 \leq a \leq p\,q - 1 \mid q \mid a\}}_{p-1 \text{ elements}} \right]$.
Hence: $\varphi(p\,q) = (p\,q - 1) - (q-1) - (p-1) = p\,q - p - q + 1 = (p-1)(q-1) = \varphi(p)\,\varphi(q)$.

**d)** Apply the Euler phi-function on $n$ with the following steps:

1. Factorize all prime factors of the given $n$
2. Apply the rules in a) to c), correspondingly.

$\varphi(4913) = \varphi(17^3) \overset{(b)}{=} 17^2(17-1) = 4624$, and
$\varphi(899) = \varphi(30^2 - 1^2) = \varphi((30-1)(30+1)) = \varphi(29 \cdot 31) \overset{(c)}{=} 28 \cdot 30 = 840$.