Homework 7 in Optimization in Engineering

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Exercise 1. (conjugate functions) The *conjugate function* of $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined as

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x}\in \operatorname{dom} f} \left\{ \boldsymbol{y}^T\boldsymbol{x} - f(\boldsymbol{x}) \right\}.$$

The domain of f^* consists of all $\boldsymbol{y}$ with $f^*(\boldsymbol{y}) < \infty$. For the functions below, compute a closed-form expression for the conjugate function and describe its domain.

a) $f(x) = e^x$, $x \in \mathbb{R}$

b) $f(x) = x \log x$, $x > 0$

c) $f(x) = \frac{1}{x}$, $x > 0$

d) $f(\boldsymbol{x}) = \log\left(\sum_{i=1}^n e^{x_i}\right)$ with $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$.

For **d)**, assume that $\operatorname{dom} f^* = \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y} \geq \boldsymbol{0},\ \sum_{i=1}^n y_i = 1\}$ is known.

Exercise 2. (log-concavity) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable, $\operatorname{dom} f$ is convex, and $f(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \operatorname{dom} f$. Show that f is log-concave if and only if for all $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} f$,

$$\frac{f(\boldsymbol{y})}{f(\boldsymbol{x})} \leq \exp\left(\frac{\nabla f(\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x})}{f(\boldsymbol{x})} \right).$$

Exercise 3. (optimal sets and values) Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(x_1, x_2) \\ \text{subject to} \quad & 2x_1 + x_2 \geq 1, \quad x_1 + 3x_2 \geq 1, \quad x_1 \geq 0, \quad x_2 \geq 0\,. \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

a) $f_0(x_1, x_2) = x_1 + x_2$

b) $f_0(x_1, x_2) = -x_1 - x_2$

c) $f_0(x_1, x_2) = x_1$

d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$

e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$