Homework 12 in Optimization in Engineering

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11.07.2012

Exercise 1. (barrier method) Let

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & 2 \leq x \leq 4 \end{array}$$

be an optimization problem with $f(x) = x + 1$, $x \in \mathbb{R}$. The feasible set is $[2, 4]$ and the optimal solution $x^* = 2$. Formulate the logarithmic barrier function $\Phi(x)$ and calculate the optimal solution $x^*(t)$ of the problem

$$\text{minimize} \quad tf(x) + \Phi(x)$$

with $x \in \mathbb{R}$ and constant $t > 0$. Illustrate the development of $x^*(t)$ and $f(x^*(t))$ for increasing t. What happens for $t \to \infty$?

Exercise 2. (inside and outside balls) We consider the following simplex, which is related to Karmarkar's standard form for linear problems

$$\Sigma = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \mathbf{1}^T \boldsymbol{x} = n, \boldsymbol{x} \geq 0 \right\} ,$$

with $\boldsymbol{x} \in \mathbb{R}^n$, $n \geq 2$. Furthermore let $B_r(\boldsymbol{x}_c)$ denote the $(n-1)$-dimensional euclidian ball with center $\boldsymbol{x}_c \in \mathbb{R}^n$ and radius $r \geq 0$, which lies in the $(n-1)$-dimensional (affine) space formed by Σ.

Prove the following statements.

a) The smallest ball at $\boldsymbol{x}_c = \mathbf{1}$, that contains Σ (i.e. outside ball), has radius

$$r = \sqrt{n(n-1)} \, .$$

b) The largest ball at $\boldsymbol{x}_c = \mathbf{1}$, that is contained in Σ (i.e. inside ball), has radius

$$r = \sqrt{\frac{n}{n-1}} \, .$$

Hint: Formulate the optimization problems that calculate the maximum and minimum distance between points in the simplex and the center of the ball. Then evaluate the KKT conditions. In b) limit the calculation to the boundary

$$\partial\Sigma = \{ \boldsymbol{x} \in \Sigma \,|\, \exists\, i \in \{1, \ldots, n\} \colon x_i = 0 \}$$

of Σ.