# Homework 2 in Optimization in Engineering

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**Exercise 1.** (Convex sets and figures) Show that the following sets are convex.

(a) The set $\mathcal{C} = \bigcap_{i \in \mathcal{I}} \mathcal{C}_i$ as intersection of convex sets $\mathcal{C}_i$ where $\mathcal{I}$ is an index set.

(b) A slab $\{\boldsymbol{x} \in \mathbb{R}^n \mid \alpha \leq \boldsymbol{a}^T \boldsymbol{x} \leq \beta\}$ with $\boldsymbol{a} \in \mathbb{R}^n_{\neq 0}$ and $\alpha, \beta \in \mathbb{R}$.

(c) A rectangle $\{\boldsymbol{x} \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, n\}$ with $\alpha_i, \beta_i \in \mathbb{R},\ 1 \leq i \leq n$.

(d) A wedge $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_1^T \boldsymbol{x} \leq \beta_1,\ \boldsymbol{a}_2^T \boldsymbol{x} \leq \beta_2\}$ with $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^n_{\neq 0}$ and $\beta_1, \beta_2 \in \mathbb{R}$.

**Hint:** Halfspaces $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^T \boldsymbol{x} \leq b\}$ with $\boldsymbol{a} \in \mathbb{R}^n_{\neq 0}$ and $b \in \mathbb{R}$ are convex sets.

**Exercise 2.** (Polyhedron) Which of the following sets $\mathcal{S} \subseteq \mathbb{R}^n$ describe a polyhedron? Describe, if possible, the set as $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}\}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n},\ \boldsymbol{C} \in \mathbb{R}^{p \times n},\ \boldsymbol{b} \in \mathbb{R}^m,\ \boldsymbol{d} \in \mathbb{R}^p$.

(a) $\mathcal{S} = \{y_1 \boldsymbol{a}_1 + y_2 \boldsymbol{a}_2 \mid -1 \leq y_1 \leq 1,\ -1 \leq y_2 \leq 1\}$ with $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^n$.

(b) $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0},\ \sum_{i=1}^n x_i = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$ with $a_1, \dots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0},\ \boldsymbol{x}^T \boldsymbol{y} \leq 1 \text{ for all } \boldsymbol{y} \text{ with } \|\boldsymbol{y}\|_2 = 1\}$.

(d) $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0},\ \boldsymbol{x}^T \boldsymbol{y} \leq 1 \text{ for all } \boldsymbol{y} \text{ with } \|\boldsymbol{y}\|_1 = 1\}$.

**Exercise 3.** (Semidefinite matrices and cones)

(a) Show that the eigenvalues of a positive semidefinite matrix are nonnegative.

(b) Prove the following equivalence for the positive semidefinite cone in $\mathcal{S}^2$.

$$\boldsymbol{X} = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathcal{S}^2_{\geq 0} \iff x \geq 0, z \geq 0, xz \geq y^2.$$