Homework 11 in Optimization in Engineering

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Exercise 1. (Optimality conditions) Consider the optimization problem

$$\begin{aligned} \text{minimize } & x_1^2 + x_2^2 \\ \text{subject to } & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \end{aligned}$$

with variable $\boldsymbol{x} \in \mathbb{R}^2$.

(a) Sketch the feasible set and level sets of the objective. Find the optimal point $\boldsymbol{x}^*$ and the optimal value p^*.

(b) Give the expression of the associated Langrangian and state the KKT conditions. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove that $\boldsymbol{x}^*$ is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

Exercise 2. (Gradient descent method with exact line search) The algorithms for unconstrained optimization problems in the lecture produce a minimizing sequence $\{\boldsymbol{x}^{(k)}\}_{k\in\mathbb{N}}$ where

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + t^{(k)} \Delta \boldsymbol{x}^{(k)}$$

and $t^{(k)} > 0$. The vector $\Delta \boldsymbol{x}$ is called the *step*, the scalar $t^{(k)}$ the *step size*, or *step length*. The methods discussed in the following are *descent methods* which means that

$$f(\boldsymbol{x}^{(k+1)}) < f(\boldsymbol{x}^{(k)}),$$

except when $\boldsymbol{x}^{(k)}$ is optimal.

A general descent method alternates between two steps, determining a descent direction $\Delta \boldsymbol{x}$, and the selection of a step size t. The natural choice for the search direction is the negative gradient $\Delta \boldsymbol{x} = -\nabla f(\boldsymbol{x})$. The resulting algorithm is called *gradient descent method*. The step size in exact line search is determined by

$$t = \operatorname{argmin}_{s>0} f(\boldsymbol{x} + s\Delta \boldsymbol{x}),$$

in which t is chosen to minimize f along the ray $\{\boldsymbol{x} + s\Delta \boldsymbol{x} \mid s \geq 0\}$.

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Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the quadratic function

$$f(\boldsymbol{x}) = \frac{1}{2}\left(x_1^2 + \gamma x_2^2\right)$$

with $\gamma > 0$. Show that the minimization of f using gradient descent method with exact line search and starting point $x^{(0)} = (\gamma, 1)$ leads in the k^{th} iteration to

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k ,$$

$$x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k .$$