Prof. Dr. Anke Schmeink, Ehsan Zandi, Yulin Hu

Tutorial 2

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Problem 1. (Semidefinite matrices and cones)

a) Show that the eigenvalues of a positive semidefinite matrix are nonnegative.

b) Prove the following equivalence for the positive semidefinite cone in $\mathcal{S}^2$.

$$\boldsymbol{X} = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathcal{S}^2_{\geq 0} \Longleftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$

Problem 2. (Convexity of norms) Let us define $\mathcal{S}_k$ as follows:

$$\mathcal{S}_k = \{\boldsymbol{x} \,|\, \boldsymbol{x} \in \mathbb{R}^n, \, ||\boldsymbol{x}||_k \leq 1\} \,, \quad k \in \mathbb{N}.$$

Now determine the convexity of the following sets using the definition of convexity.

(Hint: Using triangle inequality for norms is useful)

a) $\mathcal{S}_k, \forall k \in \mathbb{N}$

b) $\mathbb{R}^n \setminus \mathcal{S}_k, \forall k \in \mathbb{N}$

c) $\mathcal{S}_k \cap \mathcal{S}_m, \forall m, k \in \mathbb{N}$

d) $\mathcal{S}_k \cup \mathcal{S}_m, \forall m, k \in \mathbb{N}$

e) $\mathcal{S}_k - \mathcal{S}_m, \forall m, k \in \mathbb{N}$

f) $\mathcal{S}_\infty \cap \mathcal{C}, \mathcal{C} = \left\{\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \mathbb{R}^n, \sum_{m=1}^{\infty} \frac{||\boldsymbol{x}||_m^m}{m!} \leq 2n\right\}$

Problem 3. (Function convexity) Find values of a such that all following functions are convex (or concave) with a unique global minimum (or maximum).

Hint: We are not interested in monotonic functions, even though they have both global minimum and maximum for a bounded domain. One main reason from optimization point of view is that maximizing (or minimizing) monotonic functions is trivial.

a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^4 - 6a^2x^2 + 1, \quad -1 \leq x \leq 1.$

b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x - ae^{-x}, \quad -\ln m \leq x \leq \ln m, m > 1.$

c) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x + \ln(x^2 + a^2), \quad -0.5 \leq x \leq 0.5.$